THE STOCHASTIC COMPRESSIBLE NAVIER–STOKES SYSTEM ON THE WHOLE SPACE AND SOME SINGULAR LIMITS

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Abstract

Firstly, we show the existence of at least one non-trivial solution to the stochastically forced compressible Navier–Stokes system defined on the whole Euclidean space. This solution is deterministically weak in the usual sense of distributions but also weak in the sense of probability, the latter meaning that the underlying probability space, as well as the stochastic driving force, are also unknowns.

Secondly, we study various asymptotic results for the above mentioned system when the microscopic time and space variables are rescaled appropriately. Different rescaling leads to various singular versions of this system with coefficients which either blow up or dissipate when they are made small. Subsequently, we are able to show that any family of the solutions constructed above parametrised by the singular coefficients converges to solutions of other fluid dynamic models like the incompressible Navier–Stokes system and the compressible Euler system with corresponding stochastic forcing terms. Crucially, we also consider the case when rotation in the fluid is taken into account.

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Contents

Notations

We combine notations from the books [10], [15] and the paper [87].

- c, C are generic constants that may differ from line to line,
- $c_{p,s} = c(p, s)$ is a constant that depends on the parameters p and s,
- $a \lesssim b$ represents $a \leq c\, b$ for a generic constant c,
- $a \lesssim_p b$ represents $a \leq c(p)\, b$ for a constant c depending on p,
- $a \sim b$ means $a \lesssim b$ and $b \lesssim a$,
- $\mathbb{R}^n$ is the n-dimensional Euclidean space,
- $\mathbb{R}_+ = [0, \infty)$
- $\mathbb{T} = \mathbb{T}_1$ is the one-dimensional flat torus of period 2 centred at the origin,
- $\mathbb{T}^n_L$ is the n-dimensional flat torus of period $2L$ centred at the origin,
- $\mathbb{T}^n$ is the n-dimensional flat torus of a fixed arbitrary period,
- $\mathbf{a} \cdot \mathbf{b}$ is the vector scaler product $\sum_i a_i b_i$
- $\mathbb{A} : \mathbb{B}$ is the matrix scaler product $\sum_{ij} A_{ij} B_{ij}$,
- $\mathbb{I}$ is the identity matrix $\delta^n_{ij=1}$ in $\mathbb{R}^{n\times n}$,
- $\mathrm{vol}(K) = \mathcal{L}^n(K)$ volume of $K \subseteq \mathbb{R}^n$,
- $\mathrm{spt}\, f = \mathrm{supp}\, f = \overline{\{x \in X \,:\, f(x) \neq 0\}}$ is the support of the function $f : X \to Y$ for a topological space (X, τ) and a normed space Y,
- B_k is a subset of $\mathbb{R}^n$ representing a closed ball with radius k centred at the origin,
- $\mathcal{O}$ is a spatial domain in or including $\mathbb{R}^n$ itself,
- $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$ are equipped with the metrics

$$d(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left\{1, \|\mathbf{u} - \mathbf{v}\|_{L^p(B_k)}\right\}$$

$$d(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left\{1, \|\mathbf{u} - \mathbf{v}\|_{W^{k,p}(B_k)}\right\},$$

- $D^{1,q}(\mathcal{O})$ is the homogeneous Sobolev space defined by

$$D^{1,q}(\mathcal{O}) = \begin{cases} \mathbf{u} \in \mathcal{D}'(\mathcal{O}) \,:\, \mathbf{u} \in L^{\frac{3q}{3-q}}(\mathcal{O}),\ \nabla \mathbf{u} \in L^q(\mathcal{O}) & \text{if } 1 \leq q < 3, \\ \mathbf{u} = \{\overline{\mathbf{u}} + c\}_{c\in\mathbb{R}} \,:\, \mathbf{u} \in L^q_{\mathrm{loc}}(\mathcal{O}),\ \nabla \mathbf{u} \in L^q(\mathcal{O}) & \text{if } q \geq 3, \end{cases}$$

- $L^p_q(\mathcal{O})$ is an Orlicz space defined by
$$L^p_q(\mathcal{O}) = \left\{u \in L^1_{\text{loc}}(\mathcal{O}) : \text{ for } \delta > 0,\, u\mathbb{1}_{\{|u|<\delta\}} \in L^q(\mathcal{O}),\, |u|\mathbb{1}_{\{|u|\geq\delta\}} \in L^p(\mathcal{O})\right\}$$
for $1 < p, q < \infty$ and endowed with the norm
$$\left\|u\mathbb{1}_{\{|u|<\delta\}}\right\|_{L^q(\mathcal{O})} + \left\|u\mathbb{1}_{\{|u|\geq\delta\}}\right\|_{L^q(\mathcal{O})},$$
- $\mathscr{S}$ is the Schwartz class of rapidly decaying smooth functions,
- $\mathscr{S}'$ is the space of tempered distributions; the dual of $\mathscr{S}$,
- $C(\mathcal{O}) = C(\mathcal{O};\mathbb{R})$ is the space of real-valued continuous functions,
- $C_b(\mathcal{O}) = C_b(\mathcal{O};\mathbb{R})$ is the space of real-valued bounded continuous functions,
- $C^k(\mathcal{O};\mathbb{R}^m)$ is the space of $k-$times differentiable functions $f : \mathcal{O} \to \mathbb{R}^m$ with continuous derivatives and endowed with the sup-norm,
- $C_c(\mathcal{O};\mathbb{R}^m)$ is the space of continuous functions with compact support,
- $\mathcal{D}(\mathcal{O};\mathbb{R}^m) = C^\infty_c(\mathcal{O};\mathbb{R}^m)$ is the space of compactly supported smooth functions or simply, the space of *test functions*,
- $\mathcal{D}'(\mathcal{O};\mathbb{R}^m)$ is the dual space of $\mathcal{D}(\mathcal{O};\mathbb{R}^m)$ consisting of distributions,
- $C_0(\mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity,
- $\mathcal{M}(\mathbb{R}^n)$ is the space of signed Radon measures with finite mass. It is the dual of $C_0(\mathbb{R}^n)$,
- $C^\infty_{c,\text{div}}(\mathbb{R}^n) := \{f \in C^\infty_c(\mathbb{R}^n) \,:\, \operatorname{div} f = 0\}$,
- $L^2_{\text{div}}(\mathbb{R}^n) = \overline{C^\infty_{c,\text{div}}(\mathbb{R}^n)}^{\|\cdot\|_{L^2}}$ and analogously for Sobolev spaces,
- $C([a,b];X)$ is the space of continuous functions on $[a,b]$ taking values in the Banach space X and endowed with the norm
$$\|f\| = \sup_{t\in[a,b]} \|f\|_X,$$
- $C^\alpha([a,b];X)$ is the space of α-Hölder continuous functions on $[a,b]$, $\alpha \in (0,1]$, taking values in the Banach space X and endowed with the norm
$$\|f\|_{C^\alpha} = \sup_{t\in[a,b]} \|f\|_X + \sup_{\substack{s,t\in[a,b]\\ s\neq t}} \left\{\frac{\|f(t)-f(s)\|_X}{(t-s)^\alpha}\right\},$$
- $C_w([a,b];X)$ is the space of weakly continuous functions on $[a,b]$ taking values in X, i.e., the set of functions $u : [a,b] \to X$ for which the weak convergence
$$u(t_k) \rightharpoonup u(t) \quad \text{in} \quad X$$

holds for any sequence $(t_k) \in [a,b]$ satisfying $t_k \to t$. More precisely,

$$C_w([a,b];X) := \Big\{ u \in L^\infty(a,b;X) \, : \, \lim_{t_k \to t} \langle u(t_k), v\rangle = \langle u(t), v\rangle \\ \forall v \in X \text{ and for a.e. } t \in [a,b] \Big\}$$

- $L^\infty_{w^*}\big(\mathcal{O};\mathcal{M}(\mathbb{R}^n)\big)$ is the space of weakly$*$ measurable maps $\nu : \mathcal{O} \to \mathcal{M}(\mathbb{R}^n)$ (i.e. the map $x \mapsto \langle \nu_x, f\rangle$ are measurable for all $f \in C_0(\mathbb{R}^n)$) that are essentially bounded. It is the dual space of $L^1\big(\mathcal{O};C_0(\mathbb{R}^n)\big)$,
- for a topological space X, we write (X,w) if it is equipped with the weak topology,
- $X \times Y = \{(x,y) \, : \, x \in X \text{ and } y \in Y\}$ for function spaces X and Y,
- $\Delta^{-1}_{\mathcal{O}}$ represents the inverse of the Laplacian on $\mathcal{O}$ with respect to zero Dirichlet boundary conditions,
- $\Delta^{-1} = \Delta^{-1}_{\mathbb{R}^3}$ represents the inverse of the Laplacian on $\mathbb{R}^3$,
- $\mathcal{P}\mathbf{v}$ is Helmholtz decomposition onto solenoidal fields,
- $\mathcal{Q}\mathbf{v}$ is the gradient part of the vector $\mathbf{v}$,
- $\|\cdot\|_X$ is the norm on X,
- $\langle \cdot, \cdot \rangle$ is either the L^2 inner product or duality pairing depending on the context,
- $\rightharpoonup$ weak convergence,
- $\to$ strong convergence,
- $\mathbb{E}$ is the expectation,
- $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space,
- $(\mathscr{F}_t)_{t\geq 0}$ is a filtration or an increasing family of sigma algebras,
- $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\big)$ is a stochastic basis or filtered probability space,
- $L_2(\mathfrak{U}, H)$ is the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to H,
- $W(t)$ is an $(\mathscr{F}_t)$-cylindrical Wiener process,
- $\mathcal{B}(X)$ is the Borel σ-algebra on X,
- $t \wedge s = \min\{t,s\}$ for any $t,s \in \mathbb{R}$,
- $\langle\langle \cdot \rangle\rangle$ is the quadratic variation,
- $\langle\langle \cdot, \cdot \rangle\rangle$ is the cross variation.

Chapter 1

Introduction

The study of *fluid mechanics* is of fundamental importance in engineering, mathematics and physics. This is because of its wide ranging applications in real life such as in aerodynamics, weather forecasting, oceanography and astrophysics, amongst others. Built on the foundation of *conservation laws*, fluid mechanics helps describe the flow and interactions of gases, liquids and/or of plasmas as well as the forces acting on them. Until fairly recently, these forces have largely been considered *deterministic*. This means that these forces are functions of the *microscopic* space and time parameters so that at any given instant of time, the fluid's position in space is expected to be known. However, this description is a fairly weak idealization, evident in the fact that we are still unable to model extreme fluid mechanic events like *turbulence* to a sufficient level of accuracy. The modelling of turbulence can be seen as the prime motivation for the introduction of *stochasticity* in the study of fluids. In this case, *observables* such as the fluid's *density* and *velocity*, as well as certain forces, not only depend on time and space, but may also depend on a random parameter. So, at any given time, the position of these observables or forces are not claimed to be known explicitly, but with a certain probability.

1.1 Modelling of compressible fluid flows

Let $\mathbf{x} = (x_1, x_2, x_3)$ be an *independent variable* in Euclidean space $\mathbb{R}^3$. Let us consider a sufficiently-small fixed constant-volume *element* $\mathrm{d}V \ll 1$ in a physical domain $\mathcal{O} \subseteq \mathbb{R}^3$ in space so that the *density* $\varrho = \varrho(t, \mathbf{x})$ is uniform within it at any instant of time $t \geq 0$ and has *mass* of $\varrho\,\mathrm{d}V$. Now if there is a flow of an *isothermal* fluid, this will result in a transfer of mass in and out of this element. With a fluid *velocity* of $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, this yields a local *mass flux* or mass flow per unit area (which coincides with *momentum density* or simply *momentum*) of $\varrho\mathbf{u} = \varrho\mathbf{u}(t, \mathbf{x})$. The *net* rate of *outflow* (or correspondingly *inflow*) of mass per unit volume is then given by $\mathrm{div}(\varrho\mathbf{u})$ (or correspondingly $-\mathrm{div}(\varrho\mathbf{u})$). If we now equate the net rate of inflow of mass to the rate of accumulation in the element, we obtain

$$\partial_t\big(\varrho\,\mathrm{d}V\big) = -\mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}V$$

and subsequently,

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0 \tag{1.1}$$

after cancellation of the constant volume.

Equation (1.1) is referred to as the *mass balance equation* or the *continuity equation* and it is equivalent to

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\mathrm{div}\,\mathbf{u} = 0 \tag{1.2}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} := \partial_t + \mathbf{u}\cdot\nabla \tag{1.3}$$

denotes the *total derivative* or *material derivative.*

Now if $\mathbf{F} = \mathbf{F}(t, \mathbf{x})$ is a *deterministic* force acting on the fluid per unit volume (i.e.

per $\mathrm{d}V \equiv 1$), then by *Newton's second law of motion* which ensures the balance between forces and the product of mass and *acceleration*, we find that

$$\varrho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}. \tag{1.4}$$

However since $\varrho \partial_t \mathbf{u} = \partial_t(\varrho \mathbf{u}) - (\partial_t \varrho)\mathbf{u}$, by using the continuity equation (1.1), we observe that for (1.3), the left-hand side of (1.4) becomes

$$\begin{aligned} \varrho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= \partial_t(\varrho \mathbf{u}) - (\partial_t \varrho) \cdot \mathbf{u} + (\varrho \mathbf{u}) \cdot \nabla \mathbf{u} \\ &= \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u}) \cdot \mathbf{u} + (\varrho \mathbf{u}) \cdot \nabla \mathbf{u} \\ &= \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \end{aligned} \tag{1.5}$$

and thus, (1.4) becomes

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathbf{F}. \tag{1.6}$$

In general, forces that act on a particle may be classified as *conservative forces* like gravity and *non-conservative forces* which usually arise from *shear stress*. Conservative forces are generally of the form $\varrho \mathbf{f} = \pm \varrho \nabla G$ for some *potential* G such that $\mathrm{curl}\, \nabla G = 0$, whereas the non-conservative part is represented by $\mathrm{div}\, \mathbb{T}$ with $\mathbb{T}$ representing the *stress tensor*. That is

$$\mathbf{F} = \mathrm{div} \mathbb{T} + \varrho \mathbf{f} \tag{1.7}$$

and subsequently, (1.6) becomes

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div} \mathbb{T} + \varrho \mathbf{f}. \tag{1.8}$$

However, by *Stokes' law*, the stress tensor $\mathbb{T}$ satisfies

$$\mathbb{T} = \mathbb{S} - p\mathbb{I} \tag{1.9}$$

where $\mathbb{S} = \mathbb{S}(\nabla \mathbf{u})$ represents the *viscous stress tensor*, $p = p(\varrho)$ is the *pressure* and

$\mathbb{I}$ is the identity matrix. By following *Newton's law of viscosity* (i.e. *Newtonian viscous fluid*) or *Newton's rheological law*, we assume that $\mathbb{S}$ satisfies the identity

$$\mathbb{S}(\nabla \mathbf{u}) = \nu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \mathrm{div}\, \mathbf{u} \mathbb{I} \right) + \lambda \, \mathrm{div}\, \mathbf{u} \mathbb{I} \tag{1.10}$$

with the *shear viscosity coefficient* and the *bulk viscosity coefficient* satisfying $\nu > 0$ and $\lambda \geq 0$ respectively. These coefficients may in general depend on the density and even temperature but, unless otherwise stated, we are only interested in constant viscosity coefficients. Recall the assumption that our fluid is *isothermal.*

By collecting the information (1.1)–(1.10), we obtain the *Navier–Stokes system* of equations

$$\begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= \mathrm{div}\, \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f}, \\ \mathrm{div}\, \mathbb{S}(\nabla \mathbf{u}) &= \nu \Delta \mathbf{u} + (\lambda + \nu) \nabla \mathrm{div}\, \mathbf{u} \end{aligned} \tag{1.11}$$

for a *deterministic compressible Newtonian fluid.*

For the pressure, we suppose that it satisfies the γ-law

$$p(\varrho) = \frac{1}{\mathrm{Ma}^2} \varrho^\gamma \tag{1.12}$$

where $\mathrm{Ma} > 0$ is the *Mach number* and $\gamma > 1$ is the *adiabatic exponent.* A fluid with pressure satisfying (1.12) is referred to as *isentropic*, whereas in its general form $p = p(\varrho)$ (with no *explicit* relationship with density), it is referred to as *barotropic.*

On the other hand, if we take randomness into account (to model *turbulence*, say), then for a *well-defined* random force $\Phi \partial_t W$ taking its value in the set of possible *outcomes* Ω of a *probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, we may then replace the right-hand side of (1.4) with this additional term to obtain

$$\mathbf{F} = \mathrm{div} \mathbb{T} + \varrho \mathbf{f} + \Phi \partial_t W \tag{1.13}$$

so that now, $\mathbf{F} = \mathbf{F}(\omega, t, \mathbf{x})$ depends on the additional variable $\omega \in \Omega$. Notice that the stress tensor $\mathbb{T}$ and the conservative force $\varrho\mathbf{f}$ remain deterministic in this case, i.e. they only depend on time and space. However these aforementioned deterministic forces may well depend on the random parameter provided that Galilean invariance is preserved. Indeed, one may also verify the conditions on the stochastic forcing term under which the whole stochastic system is Galilean invariant, but this question remains outside the scope of this work.

There have been many models postulated for the stochastic forcing term. We refer to [81] and the references therein for some of these. In general, however, we will usually consider the family

$$\{W(t)\}_{t\geq 0} : \Omega \to \mathfrak{U} \tag{1.14}$$

of *measurable functions* from the set of possible outcomes Ω of a probability space to a *vector space* $\mathfrak{U}$ (usually a separable Hilbert space) while Φ will be a measurable function or operator from $\mathfrak{U}$ to some function space. The precise relevant assumptions for this work is given in Section 3.2.2.

With (1.13)–(1.14) in hand, we can repeat (1.6)–(1.11) to obtain the *stochastic compressible Navier–Stokes system*

$$\begin{aligned} \mathrm{d}\varrho + \mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}t &= 0, \\ \mathrm{d}(\varrho\mathbf{u}) + \big[\mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho)\big]\,\mathrm{d}t &= \big[\mathrm{div}\,\mathbb{S}(\nabla\mathbf{u}) + \varrho\mathbf{f}\big]\,\mathrm{d}t + \Phi\,\mathrm{d}W \end{aligned} \tag{1.15}$$

with the additional deterministic conservative force $\varrho\mathbf{f}$.

The aim of this work is to look at the situation on a 'very large' space. In particular, we are interested in the case when $\mathcal{O}$ is the whole space, i.e. $\mathcal{O} = \mathbb{R}^3$ or when it is the infinite cylinder $\mathcal{O} = \mathbb{R}^2 \times (0,1)$. This is particularly important for various applications and especially for those in which the comparative size of the fluid's domain far exceeds the speed of sound accompanying the fluid; see [42] for more details. Difficulties arise due to the lack of certain compactness tools which are available in the case of bounded or periodic domains.

Primarily, we will study the system

$$\begin{aligned}
\mathrm{d}\varrho + \operatorname{div}(\varrho\mathbf{u})\,\mathrm{d}t &= 0,\\
\mathrm{d}(\varrho\mathbf{u}) + [\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho)]\,\mathrm{d}t &= \operatorname{div}\mathbb{S}(\nabla\mathbf{u})\,\mathrm{d}t + \Phi(\varrho,\varrho\mathbf{u})\,\mathrm{d}W,
\end{aligned} \tag{1.16}$$

complemented with (1.10) on the time-space cylinder $Q_T = (0,T)\times\mathcal{O}$. A prototype for the stochastic forcing term is

$$\Phi(\varrho,\varrho\mathbf{u})\mathrm{d}W \approx \varrho\,\mathrm{d}W^1 + \varrho\mathbf{u}\,\mathrm{d}W^2 \tag{1.17}$$

where W^1 and W^2 is a pair of independent identically-distributed *cylindrical Wiener processes* (whose precise definition is given later in Section 2.4). We refer to Section 3.2.2 for the precise assumptions on the noise term and its coefficients.

So far, the above has dealt with fluids in non-rotating coordinate frames. If we ignore any randomness for a moment and return to (1.11), then such a fluid is governed by the continuity equality (1.1) and the following equivalent form of the momentum balance equation $(1.11)_2$

$$\varrho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \operatorname{div}\mathbb{S}(\nabla\mathbf{u}) - \nabla p(\varrho) + \varrho\,\mathbf{f} \tag{1.18}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ given by (1.3) is the material derivative.

Now let $\mathbf{x} = \big(x_1,x_2,x_3\big)$ be an arbitrary *position* vector of a fluid element lying in a Cartesian coordinate frame of reference *rotating* with *constant*[1] angular velocity $\boldsymbol{\varpi} = (\varpi_1,\varpi_2,\varpi_3)$. If we let $\mathbf{e}_1,\mathbf{e}_2$ and $\mathbf{e}_3$ be the usual unit vectors along the Cartesian axes, then since $\mathbf{e}_i$ has a *fixed* unit magnitude for each $i = 1,2,3$, we have that $\dot{\mathbf{e}}_i = \boldsymbol{\varpi}\times\mathbf{e}_i$ (where the 'dot' refers to differentiation with respect to time). Now since the relation

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{1.19}$$

[1] Note that in more complex oceanography and meteorology models with unusually long time scale, $\boldsymbol{\varpi}$ may be *non-constant*. For simplicity, we exclude such a singular phenomenon from this discussion.

holds, it follows by the product rule that the *total velocity* of this fluid element is

$$\mathbf{v} := \mathbf{u} + (\boldsymbol{\varpi} \times \mathbf{x}) \tag{1.20}$$

since by (1.19) and the identity $\dot{\mathbf{e}}_i = \boldsymbol{\varpi} \times \mathbf{e}_i$ we have that,

$$\dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3 = \dot{\mathbf{x}} = \mathbf{u},$$

$$x_1\dot{\mathbf{e}}_1 + x_2\dot{\mathbf{e}}_2 + x_3\dot{\mathbf{e}}_3 = \boldsymbol{\varpi} \times \mathbf{x}.$$

We also observe that for the *specific velocity* $\mathbf{u}$, the relation

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \tag{1.21}$$

yields a *specific acceleration* of

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + (\boldsymbol{\varpi} \times \mathbf{u}) \tag{1.22}$$

similar to (1.20). Finally, we can combine (1.20) and (1.22) together with the fact that $\boldsymbol{\varpi}$ is a constant to obtain a *total acceleration* of the fluid element

$$\begin{aligned} \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} &= \left[\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + (\boldsymbol{\varpi} \times \mathbf{u})\right] + \boldsymbol{\varpi} \times \mathbf{v} \\ &= \left[\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + (\boldsymbol{\varpi} \times \mathbf{u})\right] + \boldsymbol{\varpi} \times \mathbf{u} + \boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x}) \\ &= \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + 2\boldsymbol{\varpi} \times \mathbf{u} + \boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x}) \end{aligned} \tag{1.23}$$

The terms $2\boldsymbol{\varpi} \times \mathbf{u}$ and $\boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x})$ are the *Coriolis force* or *Coriolis acceleration* and *centripetal force* or *centripetal acceleration*, respectively. Note that the colloquial term *centrifugal force* that counteracts the *centripetal force* is not actually a force. However, the direct relationship between these two 'forces' means that we will frequently abuse terminology (by using the terms interchangeably) especially since we shall usually not assign any sign to $\boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x})$. Now, by using the identity

$$\boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x}) = -\frac{1}{2}\nabla|\boldsymbol{\varpi} \times \mathbf{x}|^2$$

which holds for the centripetal/centrifugal force, c.f. [89, Eq. 1.6.5–Eq 1.6.6], we can conclude that it is a *conservative* force. i.e.

$$\boldsymbol{\varpi} \times (\boldsymbol{\varpi} \times \mathbf{x}) = -\nabla \tilde{G} \tag{1.24}$$

for a potential $\tilde{G}(\mathbf{x}) = \frac{1}{2}|\boldsymbol{\varpi} \times \mathbf{x}|^2$.

Let us now assume that the angular velocity is $\boldsymbol{\varpi} = \frac{\mathbf{e}_3}{2}$. This corresponds to the physical situation of a fluid rotating around the vertical plane modulo the scaling parameter $\frac{1}{2}$. This parameter is just a convenient choice aimed at cancelling the constant in front of the Coriolis force. With (1.23) and (1.24) in hand, (1.18) becomes

$$\varrho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \varrho(\mathbf{e}_3 \times \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \nabla \tilde{G} + \varrho\, \mathbf{f} \tag{1.25}$$

or equivalently,

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho(\mathbf{e}_3 \times \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho\, \tilde{\mathbf{f}} \tag{1.26}$$

by the use of (1.5) and where $\tilde{\mathbf{f}} = \mathbf{f} + \nabla \tilde{G}$.

Equation (1.26) is the *momentum balance equation for deterministic rotating fluids*. By repeating the argument for (1.13) leading to (1.15), we can also obtain the *stochastically forced momentum balance equation for rotating fluids*

$$\begin{aligned} \mathrm{d}(\varrho \mathbf{u}) + &\big[\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho(\mathbf{e}_3 \times \mathbf{u}) + \nabla p(\varrho)\big]\, \mathrm{d}t \\ &= \big[\operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho\, \tilde{\mathbf{f}}\big]\, \mathrm{d}t + \Phi\, \mathrm{d}W. \end{aligned} \tag{1.27}$$

To study the relationships between any one of the compressible Navier–Stokes system above with different fluid dynamic models, one typically rescales the microscopic state variable (i.e. time and space) to get a corresponding system of equations with singular coefficients which either blow up or vanish once they are made small. See for example, Klein [64] and Alazard [2] and the references therein. The analyses into these fluid relationships are known as *singular limit results* and the rescaling performed yields the following

$$
\begin{aligned}
\mathrm{d}(\varrho\mathbf{u}) + &\left[\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ro}}\varrho(\mathbf{e}_3\times\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla\varrho^\gamma\right]\mathrm{d}t \\
&= \left[\frac{1}{\mathrm{Re}}\operatorname{div}\mathbb{S}(\nabla\mathbf{u}) + \frac{1}{\mathrm{Fr}^2}\varrho\,\tilde{\mathbf{f}}\right]\mathrm{d}t + \Phi\,\mathrm{d}W
\end{aligned}
\tag{1.28}
$$

for the stochastic system and

$$
\begin{aligned}
\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ro}}\varrho(\mathbf{e}_3\times\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla\varrho^\gamma \\
= \frac{1}{\mathrm{Re}}\operatorname{div}\mathbb{S}(\nabla\mathbf{u}) + \frac{1}{\mathrm{Fr}^2}\varrho\,\tilde{\mathbf{f}}
\end{aligned}
\tag{1.29}
$$

for the deterministic case.

The *Rossby number* Ro, which usually works in tandem with the *Froude number* Fr, is a dimensionless parameter which gives the ratio of inertial forces to the Coriolis force. As the Rossby number gets closer to zero, the influence of the Coriolis force dominates inertial forces. Consequently, the influence of horizontal fluid motion supersedes any vertical motion and, thus, essentially results in a planar two-dimensional flow. This is usually the case in large scale phenomena like meteorology and oceanic models.

The *Froude number* Fr which is also a dimensionless parameter measures the level of fluid stratification (i.e. Fr measures the vertical variation in the fluid's density). Fr may also be interpreted as the ratio of inertial forces to the conservative gravitational force (or to *buoyancy* which opposes gravity). When the Froude number is very small, the influence due to gravity (or buoyancy) outweighs the inertial force leading to highly stratified fluid.

The *Mach number* Ma for *subsonic flows*, already seen in (1.12), is also a dimensionless number in $(0,1)$ that gives the ratio of the fluid's velocity to the speed of sound. When the speed of sound dominates the velocity, this corresponds to the low Mach number regime in which case we expect an incompressible limit flow with essentially constant density.

Lastly, the dimensionless parameter Re is the *Reynolds number* which gives the ratio of inertial forces to the viscous force. In the high Reynold number limit, the 'chaotic' inertial forces dominate the damping effects of the viscous forces and, as such, transforms the fluid's profile from a laminar to a turbulent flow modelled by, say, the Euler equation or other inviscid type models.[2]

In order to study the existence of solutions to the systems (1.11), (1.15), (1.16) or any of the following (1.27)–(1.29) complemented with (1.1), they have to be further complemented with some initial data and boundary conditions. Some very common boundary conditions are periodic boundary conditions, no-slip boundary conditions and far field conditions on the whole space. As we intend to primarily work on $\mathbb{R}^3$, we shall always prescribe a far field condition

$$\mathbf{u} \to \mathbf{u}_\infty, \quad \varrho \to \varrho_\infty \tag{1.30}$$

as $|\mathbf{x}| \to \infty$ for specific choices of limit velocity $\mathbf{u}_\infty$ and density $\varrho_\infty > 0$ stated later in Section 3.2.3.

For further details on the above derivations (at least for the deterministic systems), the reader may refer to the classical book by Batchelor [5] or by Landau and Lifshitz [65]. The former treats the topic from a mathematical view point whereas the latter takes a more physical approach. Other good sources of information include the lecture notes by Childress [20] and those of Lions [71], as well as the chemical engineering book by Wilkes [108]. An excellent source of information on rotational fluids is the book by Pedlosky [89].

1.2 History

The existence of weak solutions to (1.11) with adiabatic exponent $\gamma \geq \frac{9}{5}$ has been shown in the fundamental book by Lions [72] with earlier annoucements in [69, 70]. The key idea builds on earlier discussions by Hoff [56] and Serre [97] about

[2] By using dimension analyses, each of the various parameters above may be given several equivalent definitions which may look completely unrelated. The reader should therefore have this remark in mind when reading other texts.

the improved regularity property enjoyed by the *effective viscous flux* (or *effective pressure*)

$$p(\varrho) - (\lambda + 2\nu)\mathrm{div}\,\mathbf{u} \tag{1.31}$$

as opposed to the actual pressure $p(\varrho)$. By combining this observation with the *renormalized continuity equation* introduced in [30], the author is able to analyse possible density oscillations. Lions' result was then extended to physically reasonable situations $\gamma > \frac{3}{2}$ by Feireisl, Novotný and Petzeltová [34, 43] by the use of the *oscillation defect measure* introduced in [34] to analyse such density fluctuations. These results [34, 43, 69, 70, 72] give a compressible analogue to the pioneering work by Leray [68] on the incompressible case and they involve Leray's concept of *weak solutions* where derivatives have to be understood in the *sense of distributions*. This concept of a solution has since become an integral part of the study of nonlinear PDEs.

Stemming from the physical motivations behind the dimensionless parameters introduced in the determinisitic system (1.29), several rigorous mathematical analyses via singular limit arguments have been carried out by several authors. The fundamental difficulty in performing these analyses are essentially twofold. Firstly, obtaining uniform bounds which are independent of the dimensionless parameters, and secondly the possible presence of oscillatory waves such as *acoustic waves* and/or *gravity waves* (not to be confused with gravitational waves in general relativity) for certain parts of the system. When such oscillatory waves are present, the corresponding system becomes singular and we are unable to gain compactness in a direct way. We refer to the books [42, 77] for further discussion on this phenomena and how to resolve this issues. These rigorous analyses were pioneered by Klainerman and Majda [63] where they studied these limiting behaviours in the context of general quasilinear hyperbolic systems.

The single scaling limit $\mathrm{Ma} \to 0$ (and setting $\mathrm{Fr} = \mathrm{Re} = 1$, $\mathrm{Ro} = \infty$ where $\frac{1}{\infty} := 0$) of (1.29) leading to the incompressible Navier–Stokes system has been extensively studied. This is referred to as the incompressible limit result or the low-Mach or zero Mach number result and was initiated on periodic domains in the work of Lions and Masmoudi [73]. Reference [73] also contained the inviscid-incompressible limit result which we shall talk about in the sequel. The low-Mach number result for (1.29) was then carried out on the whole Euclidean space by Desjardins and Grenier [27] and later for bounded domains in a joint work by all four authors mentioned above in [28]. See also [75, 74, 88] and the survey papers by Danchin [24] and Schochet [96].

The inviscid-incompressible limit of (1.29) which constitutes the combined limits $\mathrm{Ma} \to 0$ and $\mathrm{Re} \to \infty$ (and setting $\mathrm{Fr} = 1$, $\mathrm{Ro} = \infty$ where $\frac{1}{\infty} := 0$) in order to gain the incompressible Euler system, as mentioned earlier, was pioneered in the seminal work by Lions and Masmoudi [73] for well prepared data. Working on the whole Euclidean space, Masmoudi [78] later extended the result to more general initial data by using energy arguments in the spirit of Schochet [95]. Feireisl, Novotný and Petzeltová [47] then tackled the problem on exterior domains with slip boundary and far field conditions using the *relative energy inequality method* originally introduced by Dafermos [22] and later by Germain [53] in the context of fluid dynamics. See also [41].

Somewhat surprisingly, the single limit $\mathrm{Re} \to \infty$ (with $\mathrm{Fr} = \mathrm{Ma} = 1$ and $\mathrm{Ro} = \infty$) in (1.29) leading to the compressible Euler system has seen very few results. By relying on the relative energy inequality, Sueur [104] treated the deterministic case on bounded domains under the no-slip condition as well as the Navier boundary condition.

Finally, in the context of rotating fluids where $\mathrm{Ro} < \infty$, Feireisl and collaborators analyse the convergence of (1.29) in a series of papers [36, 37, 45, 46, 48]. These results were performed on an infinite cylinder in space $\mathcal{O} = \mathbb{R}^2 \times (0, 1)$ endowed with the complete slip

$$\mathbf{u} \cdot \mathbf{n}\big|_{\partial\mathcal{O}} = 0, \quad \left([\mathbb{S}(\nabla\mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}\right)\big|_{\partial\mathcal{O}} = 0 \tag{1.32}$$

and far field boundary conditions (1.30). The convergence to a two dimensional quasigeostrophic system with an artificial viscous force was first studied by Feireisl, Gallagher and Novotný [37] for (1.29) when $\mathrm{Re} = 1$, $\mathrm{Fr} = \infty$ (with $\frac{1}{\infty} := 0$) and $\mathrm{Ma} = \mathrm{Ro} \to 0$. The authors together with Gerard-Varet then generalized this result to when $\mathrm{Re} = 1$ and either $\mathrm{Ma} \to 0$ at the same rate or faster than $\mathrm{Fr} = \mathrm{Ro} \to 0$ in [36]. When the Mach number vanishes faster, the influence of rotation is stronger and they gain the two dimensional Navier–Stokes limit system. However, when the convergence rates are isotropic, they obtain in the limit, a linear system with radially symmetric solutions. In [46], the authors obtained an inviscid planar quasigeostrophic system from (1.29) under the isotropic rate $\mathrm{Ma} = \mathrm{Ro} \to 0$ and $\mathrm{Re} \to \infty$ when $\mathrm{Fr} = 1$. Finally, in [48, 45], the inviscid planar quasigeostrophic and the inviscid planer Euler systems are constructed from (1.29) (using alternative methods) when $\mathrm{Re} \to \infty$ and either $\mathrm{Fr} = \mathrm{Ro} \to 0$ at the same rate or slower than $\mathrm{Ma} \to 0$.

In recent years, there has been an increasing interest in random influences on fluid motions. It can take into account, for example, physical, empirical or numerical uncertainties and are commonly used to model turbulence in the fluid motion. See [8, 81, 82].

As far as we know, the first result on the stochastically forced compressible system is due to Tornatore and Yashima [106]. This was done in one dimension and later for a special periodic two dimensional case in [105]. The latter mostly relied on existence arguments developed by Vaĭgant and Kazhikhov [107] which treats a rather unphysical constitutive relation. In [40], a semi-deterministic approach based on results on multi-valued functions is used and follows in line with the incompressible analogue shown by Bensoussan and Temam in [7]. A fully stochastic theory was then developed by Breit and Hofmanová [16]. Based on the existence of *weak martingale solutions* - the definition of which we make precise later - shown by Flandoli and Gatarek [49] for the stochastic incompressible Navier–Stokes system, Breit and Hofmanová [16] show the equivalent result for the compressible counterpart under periodic boundary conditions. This has since been extended to bounded domains under Dirichlet boundary conditions by Smith [100].

Compared to the stochastic compressible model, the incompressible system which assumes constant density has been studied more intensively. It first appeared in the seminal paper by Bensoussan and Temam [7] which is based on a semi-deterministic approach. This semi-deterministic approach where the stochastic Navier–Stokes system is reduced to a system of random PDEs was then continuied by several authors including [40, 106, 105]. As already mentioned, the concept of a martingale solution of this incompressible system was later introduced by Flandoli and Gatarek [49]. For a recent survey of the stochastic incompressible Navier–Stokes equations, we refer the reader to [92] or to [67, 91] for the general survey including deterministic results.

Singular limits for stochastically forced fluids (1.28) remain largely open. The analysis for the low-Mach number limit result (where $\mathrm{Ro} = \mathrm{Fr} = \infty$, $\frac{1}{\infty} := 0$, $\mathrm{Re} = 1$ and $\mathrm{Ma} \to 0$) was only carried out recently by Breit, Feireisl and Hofmanová [11] on the torus. The drawback to this result, however, is that it could only be performed for a linear noise coefficient due to the presence of highly oscillatory acoustic waves. In [12], the same authors then studied the simultaneous limit $\mathrm{Ma} \to 0$ and $\mathrm{Re} \to \infty$ (and setting $\mathrm{Fr} = \mathrm{Ro} = \infty$) of (1.28) leading to the *incompressible* Euler limit system. Finally, a result on rotating fluids was studied by Flandoli and Mahalov [50]. Unfortunately, the original system in this case is incompressible leaving the analyses of the corresponding compressible system open. Besides these results, further open questions involve the analyses of the full system (where $\mathrm{Fr} \neq \infty$ or $\mathrm{Ro} \neq \infty$) as well as attempting to construct the *compressible* Euler system from (1.28), amongst others. There also remain open problems for more general noise coefficients and on other spatial geometries.

1.3 Main Results

This thesis contains four main results. We shall now give a summary of these results and their methods of proof. The first of these results is the following:

Existence of martingale solutions on the whole space

Here, we construct a solution to (1.16) on the whole space $\mathcal{O} = \mathbb{R}^3$, thereby extending the periodic result of Breit and Hofmanová [16]. The statement of the main result is Theorem 3.2.12. Firstly, these solutions are *deterministically weak* so that they solve (1.16) in the sense of distributions. Secondly, they are *weak in the sense of probability* meaning that the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as well as the stochastic driving force W are also unknowns.

Working on the whole space is crucial although 'realistic' physical phenomena are modelled on bounded domains. This is because when the size of these bounded domains are far larger than the speed of sound of the accompanying fluid, most of the usual *singular* fluid phenomena such as accumulation of wave fronts at the boundary are absent at any finite lapse of time. This gives the physical motivation for replacing such domains with the whole space. Furthermore, there are significant advantages (albeit some serious disadvantages like issues with embedding theorems) for working on the whole space from a purely mathematical point of view. Crucially, the well developed theory of Fourier analyses comes in handy when one considers this geometry.

The main idea of the proof is to approximate the problem on the whole space by an increasing sequence of periodic problems. We then establish uniform bounds for the random variables solving these periodic problems and then apply stochastic compactness methods to the underlying probability laws of these random variables instead of the random variables themselves. For our system (1.16), we shall deal with weak topologies of Banach spaces which are not *Polish*. Thus our compactness argument leading to the almost sure identification of our limit system on $\mathbb{R}^3$, will rely on the more general *Jakubowski–Skorokhod theorem* [57] instead of the more familiar *Skorokhod's representation theorem* for Polish spaces. As we have to deal with an unbounded domain, some refinements of the standard procedure are necessary. Eventually, we have to show strong convergence of the density in order to pass to the limit in the nonlinear pressure term. This is done by some modification of the deterministic method in [43]. In particular, we perform a stochastic adaptation of the analyses into the *effective viscous flux* (1.31) and the *oscillation defect measure* which eventually helps identify the pressure.

Incompressible limit result

As mentioned in the history section above, the low-Mach or incompressible limit result for the system (1.16) was recently shown on the torus by Breit, Feireisl and Hofmanová [11]. They established the *convergence in law* to the incompressible Navier–Stokes system. However, in identifying the limit system, they are encountered with the presence of *acoustic waves* generated in the system. These are high frequency oscillatory waves which are supported by the gradient part of velocity (and hence of momentum) and whose interactions with boundaries in bounded domains causes fundamental analytic problems. As the accompanying *acoustic waves* do not have enough ‘geometric space’ relative to the speed of sound to eventually dissipate, and thus persist, the underlining velocity vector only converges weakly. Thus, subsequent use of stochastic compactness methods only establish convergence of the solenoidal part of the fluid velocity. As such, to pass to the limit in the noise term, the authors are only left with a choice of a linear noise coefficient. By understanding this physical restriction, we are motivated to work on the whole Euclidean space. In this case, the acoustic waves quickly redistribute the associated energy. We therefore gain *dispersive estimates* and thus are able to perform the low-Mach number result for much more general class of noise coefficients.

The main argument in the proof concerns *dispersive estimates* of *Strichartz* type which hold on the whole space. With this ingredient, we obtain strong convergence of any family of momenta. Obtaining *Strichartz estimates* for hyperbolic systems has a long history tracing back to Strichartz [102, 103] and has since been especially well developed in the harmonic analyses community. See for example, [62] by Keel and Tao, as well as by Ginibre and Velo [54]. These estimates are mostly *global* in nature with the local versions developed by Smith and Sogge [99].

Inviscid limit result

Inviscid fluids are those assumed to have zero viscosity. These fluids can therefore be considered as an approximation of the Navier–Stokes system as the Reynolds number tends to infinity. Our main result in Chapter 5, stated in Theorem 5.2.11, shows that any sequence of weak martingale solutions to the Navier–Stokes system having finite energy converges locally in time to the unique strong solution of the *compressible* Euler system as the viscosity coefficients become small. A similar strategy has been employed in [12] in order to study the inviscid-incompressible limit (where in addition, the Mach number in (1.12) being small is considered), for which the limit system consists of the *incompressible* Euler equations. However, a crucial difference to [12] is that we have a much more relaxed assumption on the noise coefficients. This is not the case in the former where only linear noise coefficients may be considered due to the incompressibility constraint on the limit system.

The heart of this chapter, Section 5.3, is an application of the relative energy inequality. With this inequality, we are able to compare a weak and strong solution to two different systems, i.e., the compressible Navier–Stokes system and the compressible Euler system respectively. This strong solution of the stochastic compressible Euler system which exists locally in time has been shown recently in [17], whereas the existence of weak solution to the Navier–Stokes has already been discussed above. In light of this comparison, we gain strong convergence locally in time by combining a weak solution of the Navier–Stokes system with a pathwise solution (the stochastic notion of a strong solution) of the Euler system.

Incompressible limit for rotating compressible fluids

The last result presented in Chapter 6, Theorem 6.2.14, concerns the analyses of rotating fluids subject to both deterministic and stochastic forcing terms. These model large scale phenomena like ocean currents and the atmosphere. Unlike the previous chapters where we work on the whole three dimensional Euclidean domain, we shall now work on the semi-bounded spatial domain $\mathcal{O} = \mathbb{R}^2 \times (0,1)$ under prescribed *far field* and *complete slip* boundary conditions. Under suitable rescaling of the microscopic time and space variables leading to (1.28), we aim to show that the limit of any family of solutions to this system solves an incompressible two dimensional Navier–Stokes system. Here we set $\mathrm{Re} = 1$ and the solutions to (1.28) are indexed by the dimensionless parameters in such a way that in the limit, the stratification effect of fast rotation, measured by Ro and Fr, dominates the influence of the Mach number Ma. The reduction to a two-dimensional limit system is a result of the fast rotations due to the Coriolis force. This force causes an anisotropy in the behaviour of horizontal and vertical fluid motions. The horizontal motions dominate the vertical ones leading to the anticipated two-dimensional flow. The incompressibility of the limit system follows from the same reasoning as in Chapter 4.

Unlike the previous chapters, however, the passage to the limit in the convective term $\mathrm{div}(\varrho\mathbf{u} \otimes \mathbf{u})$ requires extra consideration. Ultimately, this is a consequence of the introduction of this Coriolis forcing term. Firstly, Helmholtz decomposition says that we can separate any vector-valued function into its curl-free and divergence-free parts. If we apply this decomposition to any family of momenta, we can show just as in the analyses of acoustic waves in the low-Mach result above, that the curl-free part (or equivalently, the gradient part) of the momenta vanishes in the limit. This is to be expected since acoustic waves are intimately related to the compressibility of the fluid. As such, one does expect to see this if we intend to obtain an incompressible limit system.

Unlike the previous chapters, the analyses of the remaining divergence-free part of momentum is not straightforward. Due to the anisotropy leading to a dominant horizontal fluid motion, we are led to analyse separately, the part of the remaining momentum vectors that are either dependent on or independent of the vertical x_3 coordinate. Fortunately, the vertical average of these vectors (meaning that they no longer depend on x_3) behaves like the two-dimensional version of the *full* divergence-free/ solenoidal part of momentum and thus, converges strongly just like in the low-Mach number result. Unfortunately, however, we are unable to establish any control on the remaining part of this solenoidal momentum that depends on x_3. Consequently, the full momentum may only converge weakly, and since the convective term depends on the momentum one will expect that it also converges weakly at best. Fortunately, we are able to show that although we have no control on this vertical dependent component, it does not constructively interfere with the nonlinear convective term. This is similar to the deterministic case, see [36], and also similar to the analyses of incompressible rotating fluids [51].

Finally, since this fast rotation has led to a two-dimensional limit system which is known to have a unique strong solution, we apply a generalization of Gyöngy–Krylov's characterization of convergence in probability [55] to identify the limit system.

1.4 Outline

Before presenting the first main result of this work, we collect in Chapter 2 some fundamental tools from analyses used at various points throughout this document. The first three sections, Sections 2.1–2.3 of that chapter, will involve analytic tools in the study of mainly deterministic PDEs. We then collect in Section 2.4, some definitions and results when randomness is taken into account in the study of PDEs. Lastly, we state the existence of Young measures and their application in deriving weak limits of Carathéodory functions, when these functions are defined on a probability space.

The first main result of this work, presented in Chapter 3, is the existence of a *finite energy weak martingale solution* to the stochastic compressible Navier–Stokes system. This is a solution which satisfies an energy inequality and is *weak* in both the analytic sense (meaning that the system holds in the sense of *distributions*) as well as in the probabilistic sense (meaning that the underlying probability space, as well as the driving stochastic force, are also unknowns). In Section 3.2 of that chapter, we collect further notation and definitions pertinent to the chapter in Section 3.2.1 and state the required assumptions on the stochastic forcing term, as well as the appropriate boundary condition applicable in our setting (Section 3.2.2 and Section

3.2.3 respectively). We also make precise the definition of such a solution and the statement of the main result in Definition 3.2.6 and Theorem 3.2.12 respectively. A formal derivation of the so-called renormalized continuity equation, a finer mathematical representation of the physical concept of the conservation of mass, is then given in Section 3.2.10.

The rest of Chapter 3 will then be devoted to the proof of the main result, Theorem 3.2.12. We approximate the system on the whole space by a sequence of periodic problems (where the period tends to infinity). After showing uniform a priori estimates in Section 3.3, we will follow this up by applying the stochastic compactness method based on the Jakubowski–Skorokhod representation theorem [57]. This is a crucial ingredient in our analyses as the original version by Skorokhod [98] does not apply to non-metrizable topological spaces in which our functions lie. In contrast to previous works by other authors, we adapt it to the situation on the whole space, taking carefully into account the lack of compact embeddings between Sobolev and Lebesgue spaces. In order to pass to the limit in the crucial nonlinear pressure term, which is done in Section 3.4, we use properties of the effective viscous flux (1.31) originally introduced by Lions [72] in a similar fashion as was done in [16]. Having identified the pressure in Section 3.4, we then complete the proof of Theorem 3.2.12 by identifying amongst the other terms, the stochastic force. Finally, the auxiliary *relative energy inequality* which is a property enjoyed by our constructed solution is presented in Section 3.6. Its origin dates back to the work of Dafermos [23] on the connection between the second law of thermodynamics and the continuous dependence of thermodynamic processes on their initial data, i.e. the stability property of these processes. Although the exact term 'relative energy inequality' was not used in that paper but was only coined later, the quote [23]

> "…estimate the evolution in time of the distance between the states of two processes originating at neighbouring states…"

very much defines this term as it is used in the current literature. It is now a fundamental and adaptable tool in the general study of *hydrodynamic limits* of which a primary example is the weak-strong uniqueness of solutions to fluid system introduced by Germain [53]. See also [44]. That is, given a weak solution of an hydrodynamic system, if a strong solution should exist, then it has to coincide or approach the weak solution, or vice versa. Other uniqueness results may also include comparison of smooth solutions with other classes of solution like measure-valued solutions [38, 26]. However, its application goes beyond uniqueness results and includes singular limit results for systems [93, 104], convergence of numerical approximations [58] and general stability results [66, 83]. See also [39] for an explicit formula for the isentropic, isothermal, compressible Navier–Stokes and Euler systems. Finally, its application to stochastic systems was first introduced in [12] for periodic solutions.

The relative energy inequality will be used as a crucial tool in the study of singular limit results in subsequent chapters.

Having established the existence of a solution to the stochastic compressible Navier–Stokes system in Chapter 3, we then proceed to establish the low-Mach number limit result on the whole space. This result seeks to identify the stochastic incompressible Navier–Stokes system as the limit of any family of weak martingale solutions of a rescaled version of the compressible system. The outline of this chapter is as follows. After a short introduction and further preliminary information including the statement of the main result in Sections 4.1–4.2, we then devote the rest of the chapter to the proof of Theorem 4.2.9. Section 4.3 will be devoted to the establishment of uniform a priori bounds for relevant families of functions with the aim of obtaining compactness results in a later section. As the low-Mach number result aims to derive an incompressible limit system, one main point will be to show that any family of densities converges to a constant in a suitable topology. Section 4.4 is devoted to showing this. A rigorous justification of the eventual dissipation of acoustic waves is established in Section 4.5 after which we show compactness of several relevant quantities in Section 4.6. Finally, we identify the incompressible Navier–Stokes system as the limit in Section 4.7. The crucial part of this section will be the proof of Proposition 4.7.6 which is a consequence of the analyses of the acoustic waves in

Section 4.5. This proposition states that the gradient part of momentum converges *strongly* to zero. This result together with the strong convergence of densities to a constant studied in Section 4.4 finally justifies the incompressibility of the limit system.

As in the earlier chapters, after a preliminary section, Section 5.2, consisting of further assumptions relevant to the chapter, definitions of relevant solutions as well as the presentation of the main result, Theorem 5.2.11, we will then devote the remaining section of Chapter 5, Section 5.3 to the proof of the main result. The main essence of this latter section, and indeed the whole chapter, will be the application of the relative energy inequality, which was introduced in Section 3.6 and further discussed above, to fully construct the limit variables solving the Euler system from the family of solutions to the Navier–Stokes system.

As mentioned earlier, we work on the new geometry $\mathcal{O} = \mathbb{R}^2 \times (0,1)$ in Chapter 6. Due to this, we will introduce some further assumptions, state the required boundary conditions, the corresponding concept of a solution and finally, the statement of the main result, Theorem 6.2.14, in Section 6.2. We will then devote the rest of the chapter to the proof of this theorem.

As usual, we start the proof of Theorem 6.2.14 by establishing various uniform estimates from which we gain compactness for the required families of functions. However, the introduction of a Coriolis term leads us to analyse its behaviour in Section 6.3.7. Having completed the compactness result, we then initiate the identification of the limit system in Section 6.3.13. Again, unlike the previous chapters, a consequence of the Coriolis term is a further complication in the mathematical analyses of the family of momenta. We thus devote Section 6.4 to the study of the momentum sequence.

Chapter 2

Preliminaries

In this chapter, we collect some fundamental tools in analysis, PDEs and probability that are required in this work. Most results will be stated without proof but with a reference to the proof stated for the interested reader.

2.1 Fourier Analysis

Let $x \in \mathbb{R}^N$ and $u \in L^1(\mathbb{R}^N;\mathbb{C})$. Then we define the Fourier transform of u as the bounded continuous function in $\mathbb{R}^N$ given by

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^N} u(x)e^{-ix\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^N. \tag{2.1}$$

If $\hat{u}(\xi)$ is also integrable, then we obtain the inverse Fourier transform

$$(\mathcal{F}^{-1}u)(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \hat{u}(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi. \tag{2.2}$$

Now if we let

$$\mathscr{S}(\mathbb{R}^N) = \left\{\varphi \in C^\infty(\mathbb{R}^N;\mathbb{C}) \,:\, \sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty, \,\forall \text{ multiindices } \alpha, \beta\right\}$$

be the *Schwartz class of rapidly decaying smooth functions*, then $\mathscr{S}(\mathbb{R}^N) \supset C_c^\infty(\mathbb{R}^N)$ is a Fréchet space with the seminorm and metric given by

$$[\varphi]_{\alpha,\beta} = \sup_x |x^\beta \partial^\alpha \varphi(x)|, \quad d(\varphi,\phi) = \sum_{\alpha,\beta} 2^{-|\alpha|-|\beta|} \frac{[\varphi-\phi]_{\alpha,\beta}}{1+[\varphi-\phi]_{\alpha,\beta}}$$

respectively. Its dual $\mathscr{S}'(\mathbb{R}^N)$ is the space of *tempered distributions* and it goes without saying that every tempered distribution is a distribution.

Remark 2.1.1. Introducing $\mathscr{S}$ is important because $C_c^\infty(\mathbb{R}^N)$ is not mapped by $\mathcal{F}$ into itself. However it does for $\mathscr{S}$ and in particular, if we let $D_j = i\partial_j$, then we have that $D_j\mathscr{S} \subset \mathscr{S}$, $x_j\mathscr{S} \subset \mathscr{S}$ and $\mathscr{S} \subset L^1$.

With Remark 2.1.1 in mind, one can extend the Fourier transform (2.1) and its inverse (2.2) as bounded linear operators from $\mathscr{S}(\mathbb{R}^N)$ into itself. Furthermore, for a tempered distribution $T \in \mathscr{S}'$, its Fourier transform can by defined by duality $\big(\mathcal{F}T\big)(\varphi) := T\big(\mathcal{F}\varphi\big)$ for $\varphi \in \mathscr{S}$.

From the above definitions, we can recall some of the fundamental properties of Fourier transforms. If φ and ϕ are in $\mathscr{S}$, then

1. Fourier transform converts differentiation into multiplication

$$\widehat{D_j\varphi} = \xi_j\widehat{\varphi}, \quad \widehat{x_j\varphi} = -D_j\widehat{\varphi}; \tag{2.3}$$

2. Fourier transfor converts convolution into multiplication

$$\widehat{\varphi * \phi} = \widehat{\varphi}\widehat{\phi}, \quad \widehat{\varphi\phi} = \frac{1}{(2\pi)^N}\widehat{\varphi} * \widehat{\phi}; \tag{2.4}$$

3. the following 'commutativity' rule applies

$$\int_{\mathbb{R}^N} \widehat{\varphi}\phi \,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi\widehat{\phi}\,\mathrm{d}x; \tag{2.5}$$

4. $(2\pi)^{-N/2}\mathcal{F}$ can be extended to a bijective isometry from $L^2(\mathbb{R}^N)$ into itself.

This gives rise to the *Parseval's formula* or the *Plancherel's identity*

$$\int_{\mathbb{R}^N} \varphi\overline{\phi}\,\mathrm{d}x = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \widehat{\varphi}\overline{\widehat{\phi}}\,\mathrm{d}\xi; \tag{2.6}$$

5. Fourier transform maps Gaussians into Gaussians. i.e.

$$\mathcal{F}\left(\left(\frac{c}{2\pi}\right)^{\frac{N}{2}} e^{\frac{-c|\cdot|^2}{2}}\right)(\xi) = e^{\frac{-|\xi|^2}{2c}} \tag{2.7}$$

   for any $c \in \mathbb{C}$ with $\mathrm{Re}(c) > 0$.

A crucial application of Fourier transform is in deriving analytic solutions to initial-valued problems on the whole space. For the purpose of this work, we demonstrate one such example.

Example 2.1.2. Let $\varphi = \varphi(t,x)$ and $\Psi = \Psi(t,x)$ be 'sufficiently smooth' functions. Then the solution of the following homogeneous initial-valued problem

$$\partial_t\varphi + \Delta\Psi = 0, \tag{2.8}$$

$$\partial_t\nabla\Psi + \gamma\nabla\varphi = 0, \tag{2.9}$$

$$\varphi(0,x) = \varphi_0(x), \quad \nabla\Psi(0,x) = \nabla\Psi_0(x), \tag{2.10}$$

for $\gamma > 0$, is given by the pair

$$\begin{aligned}
\nabla\Psi(t,x) &= \frac{e^{i\sqrt{-\gamma\Delta}t}}{2}\left(\nabla\Psi_0(x) + \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\varphi_0(x)\right) \\
&\quad + \frac{e^{-i\sqrt{-\gamma\Delta}t}}{2}\left(\nabla\Psi_0(x) - \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\varphi_0(x)\right), \\
\varphi(t,x) &= \frac{e^{i\sqrt{-\gamma\Delta}t}}{2}\left(\varphi_0(x) - \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\Psi_0(x)\right) \\
&\quad + \frac{e^{-i\sqrt{-\gamma\Delta}t}}{2}\left(\varphi_0(x) + \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\Psi_0(x)\right).
\end{aligned}$$

Proof. Using (2.3), we see that $\widehat{D_jD_j\phi} = -\widehat{\partial_j\partial_j\phi} = |\xi|^2\widehat{\phi}$. That is to say

$$\Delta \approx -|\xi|^2 \tag{2.11}$$

hence in frequency or Fourier space, the system (2.8)–(2.10) transforms into the following system of ODEs

$$\partial_t \hat{\varphi}(t,\xi) - |\xi|^2 \hat{\Psi}(t,\xi) = 0, \tag{2.12}$$

$$i\xi_i \big[\partial_t \hat{\Psi}(t,\xi) + \gamma \hat{\varphi}(t,\xi)\big] = 0, \tag{2.13}$$

$$\hat{\varphi}(0,\xi) = \hat{\varphi}_0(x), \quad i\xi_i \hat{\Psi}(0,\xi) = i\xi_i \hat{\Psi}_0(\xi). \tag{2.14}$$

By substituting (2.12) into (2.13), this is further equivalent to solving

$$\partial_t \hat{\varphi}(t,\xi) - |\xi|^2 \hat{\Psi}(t,\xi) = 0, \tag{2.15}$$

$$i\xi_i \big[\partial_t^2 \hat{\varphi}(t,\xi) + \gamma |\xi|^2 \hat{\varphi}(t,\xi)\big] = 0, \tag{2.16}$$

$$\hat{\varphi}(0,\xi) = \hat{\varphi}_0(x), \quad i\xi_i \hat{\Psi}(0,\xi) = i\xi_i \hat{\Psi}_0(\xi). \tag{2.17}$$

Now we observe that (2.16) has solutions $e^{i(\sqrt{\gamma}|\xi|)t}$ and $e^{-i(\sqrt{\gamma}|\xi|)t}$ hence we can find functions $A_\xi = A(\xi)$ and $B_\xi = B(\xi)$ which are independent of time and such that the general solution of $\hat{\varphi}$ is

$$\hat{\varphi}(t,\xi) = A_\xi e^{i(\sqrt{\gamma}|\xi|)t} + B_\xi e^{-i(\sqrt{\gamma}|\xi|)t}. \tag{2.18}$$

By invoking the initial condition (2.17), we gain

$$A_\xi + B_\xi = \hat{\varphi}_0(\xi). \tag{2.19}$$

However, by differentiating (2.18), we have that

$$\partial_t \hat{\varphi}(t,\xi) = i\sqrt{\gamma}|\xi|\, A_\xi e^{i(\sqrt{\gamma}|\xi|)t} - i\sqrt{\gamma}|\xi|\, B_\xi e^{-i(\sqrt{\gamma}|\xi|)t} = |\xi|^2 \hat{\Psi}(t,\xi) \tag{2.20}$$

where the second equality is due to (2.15). Furthermore, we can rewrite (2.20) as

$$-\sqrt{\gamma}\xi_i\, |\xi|\, A_\xi e^{i(\sqrt{\gamma}|\xi|)t} + \sqrt{\gamma}\xi_i\, |\xi|\, B_\xi e^{-i(\sqrt{\gamma}|\xi|)t} = |\xi|^2\, i\xi_i\, \hat{\Psi}(t,\xi). \tag{2.21}$$

Invoking (2.17) again then yields

$$-A_\xi + B_\xi = \frac{i|\xi|}{\sqrt{\gamma}} \hat{\Psi}_0(\xi). \tag{2.22}$$

Now solving (2.19) and (2.22) and using (2.16) gives

$$\begin{aligned} A_\xi &= -\frac{i|\xi|}{2\sqrt{\gamma}} \hat{\Psi}_0(\xi) + \frac{1}{2} \hat{\varphi}_0(\xi) \\ B_\xi &= \frac{i|\xi|}{2\sqrt{\gamma}} \hat{\Psi}_0(\xi) + \frac{1}{2} \hat{\varphi}_0(\xi) \end{aligned}$$

so that (2.15)–(2.16) becomes

$$\begin{aligned} \hat{\varphi}(t,\xi) = {} & \frac{e^{i(\sqrt{\gamma}|\xi|)t}}{2} \left(\hat{\varphi}_0(\xi) - \frac{i|\xi|}{\sqrt{\gamma}} \hat{\Psi}_0(\xi) \right) \\ & + \frac{e^{i(-\sqrt{\gamma}|\xi|)t}}{2} \left(\hat{\varphi}_0(\xi) + \frac{i|\xi|}{\sqrt{\gamma}} \hat{\Psi}_0(\xi) \right), \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} i\xi_i \, \hat{\Psi}(t,\xi) = {} & \frac{e^{i(\sqrt{\gamma}|\xi|)t}}{2} \left(-\frac{\sqrt{\gamma}}{|\xi|} \xi_i \, \hat{\varphi}_0 + i\xi_i \, \hat{\Psi}_0 \right) \\ & + \frac{e^{i(-\sqrt{\gamma}|\xi|)t}}{2} \left(i\xi_i \, \hat{\Psi}_0(\xi) + \frac{\sqrt{\gamma}}{|\xi|} \xi_i \, \hat{\varphi}_0(\xi) \right) \end{aligned} \tag{2.24}$$

respectively.

Finally, we apply inverse Fourier transform to (2.23)–(2.24) to get back to Euclidean space and the claim follows. Note that $i\xi_i \hat{u} \approx \nabla u$. □

2.2 Riesz operators and commutators

Let Δ^{-1} be the inverse Laplacian on $\mathbb{R}^3$ and consider the operator $\mathcal{A}_j : \mathscr{S}(\mathbb{R}^3) \to \mathscr{S}'(\mathbb{R}^3)$ defined by

$$\mathcal{A}_j \, u = \partial_{x_j} \Delta^{-1} u, \quad j = 1, 2, 3 \tag{2.25}$$

or in Fourier representation form:

$$\mathcal{A}_j \approx \frac{-i\xi_j}{|\xi|^2}. \tag{2.26}$$

Then the *Riesz operator* $\mathcal{R}_{ij} := \partial_{x_i}\mathcal{A}_j$ satisfies the following properties

1. $\int_{\mathbb{R}^3} \mathcal{A}_i(f)g = -\int_{\mathbb{R}^3} \mathcal{A}_i(g)f$ for any $f, g \in \mathscr{S}(\mathbb{R}^3)$;
2. $\int_{\mathbb{R}^3} u^i\mathcal{R}_{ij}v^j = \int_{\mathbb{R}^3} u^j\mathcal{R}_{ji}v^i$;
3. $\mathcal{R}_{ii}(f) = f$;
4. $\partial_j\partial_j\mathcal{A}_i(f) = \partial_i(f)$.

The verification of the above properties can directly be shown using the properties of Fourier transforms stated in Section 2.1 above. See [86, Section 4.4.1] for further details. Furthermore, these operators $\mathcal{A}_i$ are bounded linear operators from Lebesgue spaces into classes of Sobolev spaces under suitable integrability conditions. The precise relations are given in the following lemma.

Lemma 2.2.1. Let $\phi \in C_c^\infty(\mathbb{R}^3)$ and $\mathcal{A}_i$ for $i = 1, 2, 3$, be the multiplier defined in (2.25)–(2.26). Then the following holds:

$$\begin{aligned}
&\|\mathcal{A}_i[\phi\, v]\|_{W^{1,s}(\mathbb{R}^3)} \le c(s)\, \|v\|_{L^s(\mathbb{R}^3)}, \quad 1 < s < \infty,\\
&\|\mathcal{A}_i[\phi\, v]\|_{L^q(\mathbb{R}^3)} \le c(s,q)\, \|v\|_{L^s(\mathbb{R}^3)}, \quad q < \infty,\ \frac{1}{q} \ge \frac{1}{s} - \frac{1}{3}\\
&\|\mathcal{A}_i[\phi\, v]\|_{L^\infty(\mathbb{R}^3)} \le c(s)\, \|v\|_{L^s(\mathbb{R}^3)}, \quad s > 3.
\end{aligned} \tag{2.27}$$

Lemma 2.2.1 is a consequence of the combinations of Mikhin–Hörmander's Multiplier theorem (see for example, [101, Page 93, Theorem 3]) and the Marcinkiewicz multiplier theorem (see for example, [101, Page 109, Theorem 6']). The last of (2.27) follows from the second by Sobolev's embedding. For the proofs, see for example [42, Section 10.16].

As a result of Lemma 2.2.1, we may define R_{ij} for L^p-functions. Crucially, we obtain the following additional property:

1. $\int_{\mathbb{R}^3} \mathcal{R}_{ij}(f)g = \int_{\mathbb{R}^3} \mathcal{R}_{ij}(g)f$

for any $f \in L^r(\mathbb{R}^3)$ and $g \in L^{r'}(\mathbb{R}^3)$ with $\frac{1}{r} + \frac{1}{r'} = 1$ and $1 < r, r' < \infty$.

2.3 Elementary property of the cut-off T_k

We present in the section, a cut-off function and its properties which can be found in [86, Section 4.11.3.1].

For $k > 0$, let define the function $T_k : [0, \infty) \to [0, \infty)$ and its derivative by

$$T_k(t) = \begin{cases} t & \text{if } 0 \le t < k, \\ k & \text{if } k \le t < \infty. \end{cases}, \quad (T_k)'_+(t) = \begin{cases} T'_k(t) = 1 & \text{if } 0 \le t < k \\ 0 & \text{if } k \le t < \infty \end{cases} \tag{2.28}$$

so that $(T_k)'_+ \in L^\infty(\mathbb{R}) \cap C\big([0, k) \cup (k, \infty)\big)$. Then the following properties hold

$$|T_k(t) - T_k(s)| \le |t - s| \text{ for any } s, t \in [0, \infty), \tag{2.29}$$

$$|T_k(t) - t| \le t \mathbb{1}_{t \ge k}, \tag{2.30}$$

$$|t(T_k)'_+(t) - T_k(t)| \le k \mathbb{1}_{t \ge k} = T_k(t) \mathbb{1}_{t \ge k}, \tag{2.31}$$

$$|T_k(t) - T_k(s)|^{\gamma+1} \le (t^\gamma - s^\gamma)\left(T_k(t) - T_k(s)\right). \tag{2.32}$$

Furthermore,

1. If $\overline{T_k(f)} \in L^\infty(\mathbb{R}^3)$ is the weak(*) limit as $n \to \infty$ of $T_k(f_n)$, then for all $p \in [1, q)$ there exist a constant c independent of k such that

$$\begin{aligned} \left\| \overline{T_k(f)} - f \right\|_{L^p(B)} &\le \liminf_{n \to \infty} \|T_k(f_n) - f_n\|_{L^p(B)} \\ &\le \limsup_{n \to \infty} \|T_k(f_n) - f_n\|_{L^p(B)} \\ &\le c\, k^{\frac{1}{q} - \frac{1}{p}} \|f_n\|_{L^q(B)} \end{aligned} \tag{2.33}$$

   for any ball $B \subset \mathbb{R}^N$.

2. Also,

$$\|T_k(f) - f\|_{L^p(B)} \lesssim k^{\frac{1}{q}-\frac{1}{p}} \|f_n\|_{L^q(B)} \tag{2.34}$$

$$\left\|f_n \mathbb{1}_{f_n \geq k}\right\|_{L^p(B)} \lesssim k^{\frac{1}{q}-\frac{1}{p}} \|f_n\|_{L^q(B)} \tag{2.35}$$

2.4 Concepts of stochastic analysis

Since we intend for this work to be as self-contained as possible, we collect in this section, various definitions and fundamental results that would be used throughout this document. Some good references for the elementary concepts and result in this section are [60, 61, 31, 94].

2.4.1 Fundamental concepts

The triplet $(\Omega, \mathscr{F}, \mathbb{P})$ will be the standard notation for a *complete probability space* where Ω is non-empty set, $\mathscr{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is the corresponding probability measure. The probability space $(\Omega, \mathscr{F}, \mathbb{P})$ being *complete* means that if $A \in \mathscr{F}$ and $\mathbb{P}(A) = 0$, then all subsets of A lie in $\mathscr{F}$ and have probability zero. Furthermore, every probability space can be made complete or *completed* by suitably enlarging $\mathscr{F}$ hence making this classification of completeness rather redundant. When we endow $(\Omega, \mathscr{F}, \mathbb{P})$ with the *filtration* $(\mathscr{F}_t)_{t\geq 0}$ consisting of the family of non-decreasing σ-algebras $\mathscr{F}_t \subseteq \mathscr{F}$, $t \geq 0$, then the quadruplet $\big(\Omega, \mathscr{F}, \big(\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\big)$ shall be known as a *stochastic basis* or a *filtered probability space*. Furthermore, as a minimal requirement, we shall always assume that any such filtration $(\mathscr{F}_t)_{t\geq 0}$ is *complete* and *right-continuous*. The former meaning that $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathscr{F}$ as described earlier and the latter meaning that

$$\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$$

for all $t \geq 0$.

A real-valued *stochastic process* is a set of random variables

$$X = \{X(t) \,:\, t \geq 0\} = \{X(\omega, t) \,:\, t \geq 0\}$$

on $(\Omega, \mathscr{F})$ with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Here $\mathcal{B}(\mathbb{R})$ is the *Borel* σ-algebra on $\mathbb{R}$ which is the smallest σ-algebra on $\mathbb{R}$ containing all open subsets of $\mathbb{R}$. And for fixed $\omega \in \Omega$, the mapping $t \mapsto X(\omega, t)$ is called the *path* or *trajectory* of X. Furthermore, we say X is *measurable* if the mapping

$$(\omega, t) \mapsto X(\omega, t) \quad : \quad \big(\Omega \times [0, \infty)\,,\, \mathscr{F} \otimes \mathcal{B}(\mathbb{R})\big)$$

is measurable.

We now proceed to list several relevant definitions and results.

Definition 2.4.2. A real-valued stochastic process $X = \{X(t) \,:\, t \geq 0\}$ is $(\mathscr{F}_t)$-*adapted* if for every $t \geq 0$, $X(t)$ is $(\mathscr{F}_t)$-measurable.

Definition 2.4.3. A real-valued stochastic process $X = \{X(t) \,:\, t \geq 0\}$ is *progressively measurable* if for any $t \in [0, \infty)$, the mapping

$$(\omega, s) \mapsto X(\omega, s) \quad : \quad \big(\Omega \times [0, t]\,,\, \mathscr{F}_t \otimes \mathcal{B}([0, t])\big)$$

is measurable.

Definition 2.4.4. The *law* $\mathcal{L}$ of a real-valued random variable X, denoted by $\mathcal{L}[X]$, is a probability measure obtained by

$$\mathcal{L}(X) := \mathcal{L}[X](B) = \mathbb{P}\big\{\omega \in \Omega \,:\, X(\omega) \in B\big\}, \quad B \in \mathcal{B}(\mathbb{R}).$$

Definition 2.4.5. A real-valued stochastic process M is an $(\mathscr{F}_t)$-*martingale* if

- for all $t \geq 0$, $M(t)$ is $(\mathscr{F}_t)$-measurable;
- for all $t \geq 0$, $M(t)$ is integrable;
- for all $0 \leq s \leq t$, $\mathbb{E}\big[M(t) \,|\, \mathscr{F}_s\big] = M(s)$.

We now wish to give a definition of the *cross variation* between two stochastic processes. To do this however, we require the following theorem for the existence of the *quadratic variation* of a stochastic process. See [61, Section 1.4].

Theorem 2.4.6. *Let X be a continuous real-valued $(\mathscr{F}_t)$-martingale such that $\mathbb{E}|X(t)|^2$ if finite for all $t \geq 0$. Then there exists a unique stochastic process $\langle\langle X\rangle\rangle$ such that*

- *$\langle\langle X\rangle\rangle$ is $(\mathscr{F}_t)$-adapted;*
- *$\langle\langle X\rangle\rangle(0) = 0$ $\mathbb{P}$-a.s.;*
- *$\langle\langle X\rangle\rangle$ has $\mathbb{P}$-a.s. non-decreasing path;*
- *$X^2 - \langle\langle X\rangle\rangle$ is a continuous $(\mathscr{F}_t)$-martingale.*

Definition 2.4.7. We refer to the process $\langle\langle X\rangle\rangle$ constructed in Theorem 2.4.6 as the *quadratic variation* of X.

Definition 2.4.8. Let M, N be a pair of continuous real-valued $(\mathscr{F}_t)$-martingales such that $\mathbb{E}|M(t)|^2 < \infty$ and $\mathbb{E}|N(t)|^2 < \infty$ for all $t \geq 0$. Then the following quadratic variation process

$$\langle\langle M, N\rangle\rangle := \frac{1}{4}\Big(\langle\langle M + N\rangle\rangle - \langle\langle M - V\rangle\rangle\Big)$$

is called the *cross variation* of M and N.

Definition 2.4.9. A vector-valued stochastic process W is an $(\mathscr{F}_t)$-*Wiener process* if

- W is $(\mathscr{F}_t)$-adapted;
- for $\mathbb{P}$-a.e. $\omega \in \Omega$, $W(\omega, 0) = 0$;
- it has $\mathbb{P}$-a.s. continuous sample path $t \mapsto W(t)$;
- the increment $W(t) - W(s)$ is independent of $\mathscr{F}_s$ for all $0 \leq s < t$;
- the increment $W(t) - W(s)$ has normal distribution $\mathcal{N}\big(0, (t-s)\mathbb{I}\big)$ for all $0 \leq s < t$.

Remark 2.4.10. We shall always refer to a real-valued $(\mathscr{F}_t)$-*Wiener process* as a real-valued Brownian motion or simply, a Brownian motion to differentiate between vectorial-valued and scaler-valued processes.

Definition 2.4.11. A stochastic process W is an $(\mathscr{F}_t)$-*cylindrical Wiener process* if there exists a sequence of Brownian motions $(\beta_k)_{k\in\mathbb{N}}$ and a complete orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of a separable Hilbert space $\mathfrak{U}$ such that

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t)e_k, \quad t \geq 0.$$

Definition 2.4.12. Let W be an $(\mathscr{F}_t)$-cylindrical Wiener process where $\mathscr{F}_t := \sigma\big(W(s); 0 \leq s \leq t\big)$ is the *canonical* or *natural* filtration. Then we say that a filtration $(\mathfrak{F}_t)_{t\geq 0}$ such that $\mathscr{F}_t \subset \mathfrak{F}_t$ for all $t \geq 0$ is *non-anticipative* with respect to W if for all $t \geq 0$, $\mathfrak{F}_t$ is independent of $\sigma\big(W(t+s) - W(t)\big)$ for any $s > 0$.

Definition 2.4.13. An $(\mathscr{F}_t)$-*stopping time* $\mathtt{t} : \Omega \to [0,\infty]$ defined on the stochastic basis $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ is a random variable such that $\{\omega : \mathtt{t}(\omega) \leq t\} \in \mathscr{F}_t$ for any $t \geq 0$.

Definition 2.4.14. Let $p \in [1,\infty)$. We say that the family of real-valued random variable X_n on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and with values in the Banach space $(\mathcal{S}, \|\cdot\|)$ converges *in p-moment* or converges in L^p to X, denoted $X_n \to X$ in $L^p(\Omega;\mathcal{S})$, if $\lim_{n\to\infty} \mathbb{E}\big\|X_n(\omega) - X(\omega)\big\|_{\mathcal{S}}^p = 0$.

Definition 2.4.15. We say that the family of random variables X_n defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and with values in the topological space (χ, τ) converges *almost surely* to X, denoted $X_n \to X$ $\mathbb{P}$-a.s., if $\mathbb{P}\big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\big\} = 1$.

Definition 2.4.16. Let (χ, τ) be a locally convex topological space equipped with the family of semi-norms $(d_i)_{i\in I}$ where I is an indexing set. We say that the family of χ-valued random variables X_n defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ converges *in probability* to the χ-valued random variable X, denoted $X_n \to X$ in *probability* if for all $\varepsilon > 0$ and $i \in I$, we have that $\lim_{n\to\infty} \mathbb{P}\big\{\omega \in \Omega : d_i\big(X_n(\omega) - X(\omega)\big) > \varepsilon\big\} = 0$.

Definition 2.4.17. A *Tychonoff space* is a topological space (χ, τ) that is both a Hausdorff space and a completely regular space. In other words, the following three properties hold:

- (χ, τ) is a topological space;
- for any two distinct points in χ, there are disjoint open sets containing the two points respectively;
- for any point $x \in \chi$ and closed subset $S \subset \chi$ such that $x \notin S$, there exists a continuous function $f : \chi \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in S$.

Definition 2.4.18. Let (χ, τ) be a Tychonoff space with the Borel σ-algebra such that $X : \Omega \to \chi$. Then $X_n \to X$ in *law* if $\mathbb{E}\big[f(X_n(\omega))\big] \to \mathbb{E}\big[f(X(\omega))\big]$ for any $f \in C_b(\chi)$.

Definition 2.4.19. Let $Q_T := [0, T] \times \mathbb{R}^3$. A random variable $X \in \mathcal{D}'(Q_T)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *random distribution* if $\omega \mapsto \int_{Q_T} X \,\varphi \,\mathrm{d}x\,\mathrm{d}t$ is measurable for any $\varphi(t, x) \in \mathcal{D}(Q_T)$.

Definition 2.4.20. Let $Q_T := [0, T] \times \mathbb{R}^3$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis endowed with a Borel probability measure $\mathbb{P}$ and a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. Now consider the following *canonical filtration* or *natural filtration* of X up to time $\mathtt{t}$:

$$\sigma_{\mathtt{t}}(X) = \left\{ \left\{ \int_{Q_T} X \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t < a \right\} \Big|\, a \in \mathbb{R},\, \varphi \in C_c^{\infty}([0, \mathtt{t}] \times \mathbb{R}^3) \right\}.$$

Then X is $(\mathscr{F}_t)$-progressively measurable if for any $\mathtt{t} \geq 0$, we have that $\sigma_{\mathtt{t}}(X) \subset \mathscr{F}_{\mathtt{t}}$.

Remark 2.4.21. Definition 2.4.20 is equivalent to Definition 2.4.3 if the random variable X is continuous in time. See [15, Section 2.2].

Furthermore, it is sometimes useful to interpret a *random variable* $X \in L^1(Q_T)$ as a *random distribution* with values in a larger separable Hilbert space $W^{-l,2}(Q_T)$ for some $l \geq 0$. More precisely, since the compact embedding $L^1(Q_T) \hookrightarrow W^{-l,2}(Q_T)$

holds for $l > 2$ (since dimension is $1+3$), we may define any such X as a Borel random distribution defined on the Polish space $W^{-l,2}(Q_T)$ where $l > 2$.

Definition 2.4.22. A collection $\mathcal{M}$ of probability measures on the topological space $(\chi, \mathcal{A})$ is *tight* if for any given $\varepsilon > 0$, there exists a compact set $K_\varepsilon \in \mathcal{A}$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$ for all $\mu \in \mathcal{M}$.

Definition 2.4.23. We refer to a topological space (χ, τ) as a *quasi-Polish space* (also a *sub-Polish space* [15, Definition 2.1.3.] or a *Jakubowski space* [100, Definition A.1.]) if there exists a countable family $\{f_n : \chi \to [-1,1]\}_{n\in\mathbb{N}}$ of τ-continuous functionals which separate points of χ.

With the above set of definitions in hand, we can now proceed to state some 'standard' useful results. Much of the following can be found in [21, 61, 90].

2.4.24 Itô stochastic integration

To begin with, we first make sense of an Itô stochastic integral appearing in the study of SPDEs. The following standard interpretation of said integral can be found in say, [21, Section 4.2].

Let $\mathfrak{U}$ and H be two fixed separable Hilbert space and W, an $(\mathscr{F}_t)$-cylindrical Wiener process on $(\Omega, \mathfrak{F}, \mathbb{P})$ having values in $\mathfrak{U}$ (recall Definition 2.4.11). Now consider $L_2(\mathfrak{U}; H)$ - the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to H. Then for an $L_2(\mathfrak{U}; H)$-valued $(\mathscr{F}_t)$-progressively measurable stochastic process X, the following

$$\int_0^t X(s)\,\mathrm{d}W(s) = \sum_{k\in\mathbb{N}} \int_0^t X(s)e_k\,\mathrm{d}\beta_k(s) \tag{2.36}$$

stochastic integral of X with respect to W is a *well-defined continuous H-valued square integrable $(\mathscr{F}_t)$-martingale* provided that

$$\mathbb{E}\int_0^T \left\|X(t)\right\|^2_{L_2(\mathfrak{U};H)}\,\mathrm{d}t < \infty. \tag{2.37}$$

Proposition 2.4.25 (Itô isometry)**.** Let H be a separable Hilbert space. Then for any $(\mathscr{F}_t)$-adapted stochastic process X satisfying (2.37), the following equality

$$\mathbb{E}\left\|\int_0^t X(s)\,\mathrm{d}W(s)\right\|_H^2 = \mathbb{E}\int_0^t \|X(s)\|_{L_2(\mathfrak{U};H)}^2 \mathrm{d}s$$

holds for all $t \geq 0$.

Proposition 2.4.26 (Burkholder–Davis–Gundy's inequality)**.** Let H be a separable Hilbert space. Then for any $p \in (0,\infty)$, there exists a constant $c_p > 0$ such that for any $(\mathscr{F}_t)$-progressively measurable stochastic process X satisfying (2.37), the following inequality

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t X(s)\,\mathrm{d}W(s)\right\|_H^p \leq c_p\,\mathbb{E}\left(\int_0^T \|X(s)\|_{L_2(\mathfrak{U};H)}^2 \mathrm{d}s\right)^{\frac{p}{2}}$$

holds.

Proposition 2.4.27 (Kolmogorov's continuity criterion)**.** Let H be a separable Banach space and let X be an H-valued stochastic process. If there exists constants $c > 0$, $a \geq 1$, $b > 0$ such that for all $s, t \in [0,T]$, the following inequality

$$\mathbb{E}\left\|X(t) - X(s)\right\|_H^a \leq c\,|t-s|^{1+b}$$

holds, then X has a $\mathbb{P}$-a.s. Hölder continuous modification $\tilde{X}$ with Hölder exponent $\kappa \in (0, \frac{b}{a})$. Furthermore

$$\mathbb{E}\left\|\tilde{X}\right\|_{C^\kappa([0,T];H)}^a \lesssim 1$$

uniformly of the representation $\tilde{X}$.

2.4.28 Stochastic compactness and identification of limits

Our compactness arguments will primarily be based on the following theorem by Jakubowski [57, Theorem 2].

Theorem 2.4.29 (Jakubowski–Shorokhod representation theorem)**.** *Let (χ, τ) be a quasi-Polish space. Let $\left(X_n\right)_{n\in\mathbb{N}}$ be a sequence of χ-valued random variables* *for which the corresponding family of laws $(\mu_n)_{n\in\mathbb{N}}$ is tight. Then there exists a subsequence $\big(X_{n_k}\big)_{k\in\mathbb{N}}$ and χ-valued random variables $\big(Y_k\big)_{k\in\mathbb{N}}$ and Y defined on $\big([0,1],\mathcal{B}([0,1]),\mathcal{L}^1\big)$ such that for each $k\in\mathbb{N}$, the law of $\big(Y_k\big)_{k\in\mathbb{N}}$ is given by $(\mu_{n_k})_{k\in\mathbb{N}}$ and $Y_k(\omega)$ converges in χ as $k\to\infty$ to $Y(\omega)$ for a.a. $\omega\in[0,1]$.*

We now state a generalization of Gyöngy–Krylov's characterization of convergence in probability [55] due to Breit–Feireisl–Hofmanová [15, Theorem 2.10.3].

Theorem 2.4.30. *Let (χ,τ) be a quasi-Polish space. Let $\big(X_n\big)_{n\in\mathbb{N}}$ be a sequence of χ-valued random variables for which the corresponding family of laws $(\mu_n)_{n\in\mathbb{N}}$ is tight. Suppose that for every sequence $\big(X_n,X_m\big)_{n,m\in\mathbb{N}}$, there exists a subsequence $\big(X_{n_k},X_{m_k}\big)_{k\in\mathbb{N}}$ with joint law $\mu_{n_k}\times\mu_{m_k}$ which convergences weakly star to a probability measure μ on $\chi\times\chi$ supported on the diagonal $\{[x,x];x\in\chi\}$. Then there exits a χ-valued random variable X such that up to the taking of further subsequences,*

$$X_{n_k}\to X \quad \text{in} \quad \chi \quad \text{in probability.}$$

Furthermore, the law of X, being a Radon measure, is tight on χ.

The following two results on the equivalence of laws and the non-anticipativity of a filtration are an adaptation of spatially periodic problems taken from [15, Theorem 2.9.1, Lemma 2.9.3] to the whole space.

Theorem 2.4.31. *Let $X\in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N)$ $\mathbb{P}$-a.s., $X(t)=X_0$ for $t\le 0$, be a random distribution such that the following continuous operators satisfies*

$$D(X),\ \mathbf{F}(X),\ \mathbf{g}_k(X)\in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N)\quad \mathbb{P}\text{-a.s.}$$

and

$$\int_0^T\sum_{k\in\mathbb{N}}\big|\langle\mathbf{g}_k(X),\varphi\rangle\big|^2\,\mathrm{d}t<\infty\quad \mathbb{P}\text{-a.s. for any } \varphi\in C_c^\infty(\mathbb{R}^N)$$

solves the following

$$\mathrm{d}D(X) + \operatorname{div} \mathbf{F}(X)\,\mathrm{d}t = \sum_{k\in\mathbb{N}} \mathbf{g}_k(X)\mathrm{d}\beta_k, \quad D(X_0) = D_0 \tag{2.38}$$

weakly in the sense of distributions where $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is the family of real-valued Brownian motions generating the cylindrical Wiener process W in the sense of Definition 2.4.11. Now suppose that the filtration

$$\mathscr{F}_t = \sigma\bigg(\sigma_t[X] \cup \bigcup_{k=1}^{\infty} \sigma_k\big[\beta_k\big]\bigg), \quad t \geq 0,$$

is non-anticipative with respect to W and let $\tilde{X}$ be another random distribution and $\tilde{W}$, a stochastic process such that

$$\mathcal{L}[X, W] = \mathcal{L}[\tilde{X}, \tilde{W}].$$

Then $\tilde{W}$ is a cylindrical Wiener process, the filtration

$$\tilde{\mathscr{F}}_t = \sigma\bigg(\sigma_t[\tilde{X}] \cup \bigcup_{k=1}^{\infty} \sigma_k\big[\tilde{\beta}_k\big]\bigg), \quad t \in \mathbb{R},$$

is non-anticipative with respect to $\tilde{W}$ and

$$\begin{aligned}
&\mathcal{L}_{\mathbb{R}}\bigg[\int_0^T \big[\partial_t\psi\langle D(X), \varphi\rangle + \psi\langle \mathbf{F}(X), \nabla\varphi\rangle\big]\mathrm{d}t \\
&\qquad + \int_0^T \psi\langle \Phi(X), \varphi\rangle\,\mathrm{d}W + \psi(0)\langle D(X_0), \varphi\rangle\bigg] \\
&= \mathcal{L}_{\mathbb{R}}\bigg[\int_0^T \big[\partial_t\psi\langle D(\tilde{X}), \varphi\rangle + \psi\langle \mathbf{F}(\tilde{X}), \nabla\varphi\rangle\big]\mathrm{d}t \\
&\qquad + \int_0^T \psi\langle \Phi(\tilde{X}), \varphi\rangle\,\mathrm{d}\tilde{W} + \psi(0)\langle D(\tilde{X}_0), \varphi\rangle\bigg]
\end{aligned} \tag{2.39}$$

for any time and space test functions $\psi \in C_c^\infty([0,T))$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$ respectively.

Lemma 2.4.32. Let $\big(X_n\big)_{n\in\mathbb{N}}$ and X be random distributions on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\big(W_n\big)_{n\in\mathbb{N}}$ and W, cylindrical Wiener processes also defined on

$(\Omega, \mathscr{F}, \mathbb{P})$. Assume that the filtration

$$\sigma\big(\sigma_t[X_n] \cup \sigma_t[W_n]\big), \quad t \geq 0,$$

is non-anticipative with respect to W_n for every $n \in \mathbb{N}$. If the following pair of convergence

$$\begin{aligned} \langle X_n, \varphi \rangle \to \langle X_n, \varphi \rangle \quad &\text{for any } \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N) \\ W_n \to W \quad \text{in} \quad &C\big([0,T]; \mathfrak{U}_0\big) \end{aligned}$$

holds in probability, then the filtration

$$\sigma\big(\sigma_t[X] \cup \sigma_t[W]\big), \quad t \geq 0,$$

is non-anticipative with respect to W.

The following lemma and its corollary is also taken from [15, Lemma 2.1.35 and Corollary 2.1.36].

Lemma 2.4.33. Let W a cylindrical Wiener process and B be a stochastic process in $\mathfrak{U}_0$ with both processes defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathcal{L}[W]$ be the law of both W and B that is supported on $C_{\mathrm{loc}}(\mathbb{R}_+; \mathfrak{U}_0)$. Then B is a (σ_t)-cylindrical Wiener process where $\sigma_t := \sigma(B(t); 0 \leq s \leq t)$, $t \geq 0$ is the canonical filtration.

Corollary 2.4.34. Let $\big[(\sigma_t)_{t\geq 0}, (\mathscr{F}_t)_{t\geq 0}\big]$ be a pair of filtrations defined on $(\Omega, \mathscr{F}, \mathbb{P})$ where $\sigma_t := \sigma(W(t); 0 \leq s \leq t)$, $t \geq 0$ is the canonical filtration such that $\sigma_t \subset \mathscr{F}_t$ for any $t \geq 0$. If W is an (σ_t)-cylindrical Wiener process and $\mathscr{F}_t$ is non-anticipative with respect to W, then W is an $(\mathscr{F}_t)$-cylindrical Wiener process.

We now give the equivalent version of [15, Lemma 2.6.6] on the whole space.

Lemma 2.4.35. Let $\mathfrak{U} \subset \mathfrak{U}_0$ be a separable Hilbert space. Let $\big(\Omega, \mathscr{F}, \mathbb{P}\big)$ be a complete probability space. For $n \in \mathbb{N}$, let $W_n = \sum_{k\in\mathbb{N}} e_k \beta_k^n$ be an $\big(\mathscr{F}_t^n\big)$-cylindrical

Wiener process such that

$$W_n \to W \quad \text{in} \quad C\big([0,T];\mathfrak{U}_0\big) \quad \text{in probability}$$

with $W = \sum_{k\in\mathbb{N}} e_k\beta_k$.

Also for each $n \in \mathbb{N}$, let Φ_n be an $\big(\mathscr{F}_t^n\big)$-progressively measurable stochastic process belonging to $L^2\Big(0,T;L_2\big(\mathfrak{U}_0;W^{l,2}(\mathbb{R}^3)\big)\Big)$ $\mathbb{P}$-a.s. for some $l \in \mathbb{R}$ and for which,

$$\Phi_n \to \Phi \quad \text{in} \quad L^2\Big(0,T;L_2\big(\mathfrak{U}_0;W^{l,2}(\mathbb{R}^3)\big)\Big) \quad \text{in probability.}$$

Then after possible change on a measure zero set in $\Omega \times (0,T)$, we gain that

$$\int_0^{\cdot} \Phi_n \, \mathrm{d}W_n \to \int_0^{\cdot} \Phi \, \mathrm{d}W \quad \text{in} \quad L^2\big(0,T;W^{l,2}(\mathbb{R}^3)\big) \quad \text{in probability}$$

and that Φ is a progressively measurable process with respect to

$$\sigma\Bigg(\bigcup_{k=1}^{\infty} \sigma_t\big[\Phi e_k\big] \cup \sigma_t\big[\beta_k\big]\Bigg).$$

2.4.36 An Itô formula

Finally, we give an infinitesimal version of Itô lemma for stochastic processes defined on the whole space in analogy to the periodic version [15, Theorem A.4.1].

Theorem 2.4.37. *Let $\overline{r} > 0$. Let W be an $(\mathscr{F}_t)$-cylindrical Wiener process on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Let (r,s) be a pair of stochastic processes on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying*

$$\begin{aligned} \mathrm{d}r &= D^d r \, \mathrm{d}t + \mathbb{D}^s r \, \mathrm{d}W \\ \mathrm{d}s &= D^d s \, \mathrm{d}t + \mathbb{D}^s s \, \mathrm{d}W \end{aligned} \tag{2.40}$$

on the cylinder $(0,T) \times \mathbb{R}^3$ under the far field condition

$$r \to \overline{r}, \quad s \to 0$$

as $|x| \to \infty$. *Now suppose that the following*

$$(r-\overline{r}) \in C_c^\infty\big([0,T]\times\mathbb{R}^3\big), \quad s \in C_c^\infty\big([0,T]\times\mathbb{R}^3\big) \tag{2.41}$$

holds $\mathbb{P}$*-a.s. and that for all* $1 \le q < \infty$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r-\overline{r}\|^2_{W^{1,q}(\mathbb{R}^3)}\right]^q + \mathbb{E}\left[\sup_{t\in[0,T]} \|s\|^2_{W^{1,q}(\mathbb{R}^3)}\right]^q \lesssim_q 1. \tag{2.42}$$

Furthermore, assume that $D^d r$, $D^d s$, $\mathbb{D}^s r$, $\mathbb{D}^s s$ *are progressively measurable and that*

$$\begin{aligned} &D^d r,\ D^d s \in L^q\big(\Omega; L^q(0,T;W^{1,q}(\mathbb{R}^3))\big) \\ &\mathbb{D}^s r,\ \mathbb{D}^s s \in L^2\Big(\Omega; L^2\big(0,T;L_2(\mathfrak{U};L^2(\mathbb{R}^3))\big)\Big) \end{aligned} \tag{2.43}$$

and

$$\left(\sum_{k\in\mathbb{N}} |D^r s(e_k)|^q\right)^{\frac{1}{q}}, \left(\sum_{k\in\mathbb{N}} |D^s s(e_k)|^q\right)^{\frac{1}{q}} \in L^q\big(\Omega\times(0,T)\times\mathbb{R}^3\big). \tag{2.44}$$

Finally, for some $\lambda \ge 0$*, let* Q *be* $(\lambda+2)$*-continuously differentiable function such that*

$$\mathbb{E}\sup_{t\in[0,T]} \|Q^j(r)\|^2_{(W^{\lambda,q'}\cap C)(\mathbb{R}^3)} < \infty, \quad j=0,1,2. \tag{2.45}$$

Then

$$\begin{aligned} \mathrm{d}\left[\int_{\mathbb{R}^3} sQ(r)\,\mathrm{d}x\right] &= \int_{\mathbb{R}^3}\left[s\bigg(Q'(r)\,D^d r + \frac{1}{2}\sum_{k\in\mathbb{N}} Q''(r)\big|\mathbb{D}^s r(e_k)\big|^2\bigg)\right]\mathrm{d}x\,\mathrm{d}t \\ &\quad + \big\langle Q(r), D^d s\big\rangle\,\mathrm{d}t + \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^3} \mathbb{D}^s s(e_k)\,\mathbb{D}^s r(e_k)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \mathrm{d}\bigg(\sum_{k\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^3}\Big[sQ'(r)\mathbb{D}^s r(e_k) + Q(r)\mathbb{D}^s s(e_k)\Big]\,\mathrm{d}x\,\mathrm{d}\beta_k\bigg). \end{aligned} \tag{2.46}$$

2.5 Young measures for random distributions

Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded domain. Given a sequence $(\mathbf{z}_n)_{n\in\mathbb{N}} \in L^p(\mathcal{O})$ with finite moments, we are interested in the behaviour of the associated Carathéodory function $H(\cdot,\mathbf{z}_n(\cdot)) : \mathcal{O} \to \mathbb{R}$. The following fundamental theorem of Young measures for random distributions is a direct adaptation of [15, Theorem 2.8.1–Corollary 2.8.3] to domains contained in the whole space.

Lemma 2.5.1. Let $\mathbf{z}_n : \mathcal{O} \to \mathbb{R}^M$ be a sequence of random distributions (in the sense of Definition 2.4.19) on the bounded domain $\mathcal{O} \subset \mathbb{R}^N$ such that

$$\mathbb{E}\,\|\mathbf{z}_n\|_{L^p(\mathcal{O})}^p < \infty$$

holds for some $p \in (1, \infty)$. Then there exists a subsequence $\mathbf{z}_n$ (not relabelled) and another sequence $\tilde{\mathbf{z}}_n$ defined on the standard probability space $\big([0,1], \mathcal{B}([0,1]), \mathcal{L}^1\big)$ and having the same law as $\mathbf{z}_n$, as well as the existence of a Young measure ν_x on $\mathbb{R}^M$ such that for $1 < k \leq \frac{p}{q}$,

$$H\big(\,\cdot\,\tilde{\mathbf{z}}_n(\cdot)\big) \rightharpoonup \langle \nu_x, H \rangle = \int_{\mathbb{R}^M} H(x, \boldsymbol{\xi})\,\mathrm{d}\nu_x(\boldsymbol{\xi}) =: \overline{H}(x)$$

in $L^k(\mathbb{R}^N)$ $\mathcal{L}^1$-a.s. for every Carathéodory function H satisfying the inequality $|H(x, \boldsymbol{\xi})| \lesssim 1 + |\boldsymbol{\xi}|^q$ uniformly in $x \in \mathcal{O}$ for some $1 \leq q < p$.

A good source of information on Young measures is the lecture note by Müller [84].

Chapter 3

Global existence of finite energy weak martingale solutions

3.1 Introduction

The primary aim of this chapter is the construction of distributional solutions to the stochastic compressible Navier–Stokes system on the whole space. In addition to these solutions being weak in the PDE sense, they are also weak in the probabilistic sense meaning that the construction of the underlining probability space and the stochastic driving term are part of the problem.

Our strategy will follow a weak compactness argument where we pass to the limit in a sequence of approximate solutions, each of which solves a periodic problem and where periods are increasing. Existence of these corresponding solutions in the periodic setting - which was the original study into the compressible Navier–Stokes system perturbed by these general nonlinear noise [16] - involves a four layer approximation scheme that builds on the construction of weak solutions for the deterministic counterpart in [43].

3.2 Preliminaries

We collect in this section, some notations and definitions relevant to this chapter. We also give a detailed description of the solution we wish to construct in Section 3.2.4, assumptions on certain parameters and functions in the system (1.16), as well as state the main result of this chapter in Section 5.2.10. Additionally, we give a formal derivation of an important tool in the construction of these solutions - the *renormalized continuity equation* - in Section 3.2.10. Further details on this tool will be provided later.

3.2.1 Notations and definitions

In the whole of this chapter, the microscopic state variables for the quantities in (1.16) are defined on $[0,T] \times \mathcal{O}$ for $T > 0$ fixed and where $\mathcal{O} = \mathbb{R}^3$. If we let $\mathbb{R}_+ = [0,\infty)$, then the macroscopic state variable is the pair $(\varrho, \mathbf{u})$ where

$$\begin{aligned} \varrho &: (0,T) \times \mathbb{R}^3 \to \mathbb{R}_+, \\ \mathbf{u} &: (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3. \end{aligned}$$

That is, $[\varrho, \mathbf{u}] = [\varrho(t,x), (u^1(t,x), u^2(t,x), u^3(t,x))] \in \mathbb{R}_+ \times \mathbb{R}^3$. The function ϱ corresponds to the *mass density* and $\mathbf{u}$ is the *velocity* of the fluid. Furthermore, the vector-valued product function $\mathbf{m} = (\varrho\mathbf{u}) \in \mathbb{R}^3$ is the *momentum*.

Throughout this chapter, the *overline* notation on a function $f(x)$, i.e., $\overline{f(x)}$ will refer to the limit of the sequence of functions $f(x_j)$ in a suitable topology. This should not be confused with for example, the *tilde* $\tilde{f}$ or other such notations like $\hat{f}$ which will have different meanings in different contexts and explained or defined accordingly.

3.2.2 Assumptions on the stochastic force

We enforce stochasticity by considering a random force driven by an $(\mathscr{F}_t)$-cylindrical Wiener process $W = W(t)$ defined on a stochastic basis $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\big)$. Recall Definition 2.4.11. We set $\mathbf{m} = \varrho\mathbf{u}$ and assume that there exists a subset $K \Subset \mathbb{R}^3$ and some functions $\mathbf{g}_k : \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{g}_k(\cdot, \varrho, \mathbf{m}) \in C^1_0\left(K\right), \quad \text{for any } k \in \mathbb{N}, \tag{3.1}$$

for any $(\varrho, \mathbf{m}) \in \mathbb{R}_+ \times \mathbb{R}^3$, i.e. $\mathbf{g}_k$ is compactly supported in space, and in addition, satisfy the following growth conditions:

$$|\mathbf{g}_k(x, \varrho, \mathbf{m})| \leq c_k \, \left(\varrho + |\mathbf{m}|\right), \tag{3.2}$$

$$\left|\nabla_{\varrho,\mathbf{m}}\, \mathbf{g}_k(x, \varrho, \mathbf{m})\right| \leq c_k. \tag{3.3}$$

for some constant $(c_k)_{k\in\mathbb{N}} \subset [0, \infty)$ such that $\sum_{k\in\mathbb{N}} c_k^2 \lesssim 1$.

A consequence of (3.2)–(3.3) are the estimates:

$$\sum_{k\in\mathbb{N}} |\mathbf{g}_k(x, \varrho, \mathbf{m})|^2 \lesssim \varrho^2 + |\mathbf{m}|^2, \tag{3.4}$$

$$\sum_{k\in\mathbb{N}} \left|\nabla_{\varrho,\mathbf{m}}\, \mathbf{g}_k(x, \varrho, \mathbf{m})\right|^2 \lesssim 1. \tag{3.5}$$

Then if we define the map $\Phi(\varrho, \varrho\mathbf{u}) : \mathfrak{U} \to L^1(K)$ by $\Phi(\varrho, \varrho\mathbf{u})e_k = \mathbf{g}_k(\cdot, \varrho(\cdot), \varrho\mathbf{u}(\cdot))$, we can use the continuous embedding $L^1(K) \hookrightarrow W^{-l,2}(K)$ where $l > \frac{3}{2}$, to show that $\Phi(\varrho, \varrho\mathbf{u})$ is bounded uniformly in $L_2(\mathfrak{U}; W^{-l,2}(\mathbb{R}^3))$ provided $\varrho \in L^\gamma_{\text{loc}}(\mathbb{R}^3)$ and $\sqrt{\varrho}\mathbf{u} \in L^2_{\text{loc}}(\mathbb{R}^3)$. Indeed, by the use of the aforementioned embedding, we gain

$$\begin{aligned} \left\|\Phi(\varrho, \varrho\mathbf{u})\right\|^2_{L_2(\mathfrak{U}; W^{-l,2}(\mathbb{R}^3))} &= \sum_{k\in\mathbb{N}} \left\|\mathbf{g}_k(x, \varrho, \mathbf{m})\right\|^2_{W^{-l,2}(\mathbb{R}^3)} \\ &= \sum_{k\in\mathbb{N}} \left\|\mathbf{g}_k(x, \varrho, \mathbf{m})\right\|^2_{W^{-l,2}(K)} \\ &\lesssim \sum_{k\in\mathbb{N}} \left(\int_K \mathbf{g}_k(x, \varrho, \mathbf{m})\,\mathrm{d}x\right)^2. \end{aligned} \tag{3.6}$$

On the other hand, if we let $(\varrho)_K$ be the average of ϱ over the compact set K, then we can use Minkowski's inequality and (3.4) to get

$$\begin{aligned}\sum_{k\in\mathbb{N}}\left(\int_K \mathbf{g}_k(x,\varrho,\mathbf{m})\,\mathrm{d}x\right)^2 &\lesssim (\varrho)_K\int_K\sum_{k\in\mathbb{N}}\left(\varrho^{-1}\left|\mathbf{g}_k(x,\varrho,\mathbf{m})\right|^2\right)\mathrm{d}x\\ &\lesssim \int_K\left(\varrho+\varrho|\mathbf{u}|^2\right)\mathrm{d}x\\ &\lesssim \int_K\left(1+\varrho^\gamma+\varrho|\mathbf{u}|^2\right)\mathrm{d}x\end{aligned} \tag{3.7}$$

where we used the inequality $\varrho\leq 1+\varrho^\gamma$ for $\gamma>1$.

As such, the stochastic integral $\int_0^\cdot \Phi(\varrho,\varrho\mathbf{u})\mathrm{d}W$ is a well-defined $(\mathscr{F}_t)$-martingale taking value in $W^{-l,2}(\mathbb{R}^3)$.

Lastly, we define the auxiliary space $\mathfrak{U}_0\supset\mathfrak{U}$ via

$$\mathfrak{U}_0=\left\{\mathbf{u}=\sum_{k\geq1}c_ke_k\,;\quad\sum_{k\geq1}\frac{c_k^2}{k^2}<\infty\right\} \tag{3.8}$$

and endow it with the norm

$$\|\mathbf{u}\|^2_{\mathfrak{U}_0}=\sum_{k\in\mathbb{N}}\frac{c_k^2}{k^2},\quad \mathbf{u}=\sum_{k\in\mathbb{N}}c_ke_k.$$

Then it can be shown that W has $\mathbb{P}$-a.s. $C([0,T];\mathfrak{U}_0)$ sample paths with the Hilbert–Schmidt embedding $\mathfrak{U}\hookrightarrow\mathfrak{U}_0$. See [21].

3.2.3 The prescribed far field condition

Although the Euclidean space $\mathbb{R}^3$ is a boundaryless domain, a PDE posed on the entirety of $\mathbb{R}^3$ will typically be supplemented with a *far field* condition describing the evolution near infinity.

In fluid dynamics, the energy distribution of the system suggests that the resulting steady state in the far field be given by

$$\mathbf{u}\to\mathbf{u}_\infty,\quad p\to p_\infty,\quad \varrho\to\varrho_\infty$$

as $|x| \to \infty$ for the velocity, pressure and density respectively. However, since we are interested in *isentropic* fluids, it is enough to prescribe this set of conditions on just the density and velocity field. More precisely, we shall impose the following condition

$$\mathbf{u} \to 0, \quad \varrho \to \overline{\varrho} > 0 \tag{3.9}$$

as $|x| \to \infty$.

3.2.4 Concepts of solution

To continue, let us define the notions of solution that we wish to construct in this chapter.

Definition 3.2.5. If Λ is a Borel probability measure on $L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)$, then we say that

$$[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho, \mathbf{u}, W] \tag{3.10}$$

is a *weak martingale solution* of (1.16) with initial law Λ provided

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
2. W is a $(\mathscr{F}_t)$-cylindrical Wiener process;
3. the density ϱ satisfies $\varrho \geq 0$, $t \mapsto \langle \varrho(t, \cdot), \phi \rangle \in C([0, T])$ for any $\phi \in C^\infty_c(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho(t, \cdot), \phi \rangle$ is progressively measurable and
$$\mathbb{E}\left[\sup_{t \in (0,T)} \|\varrho(t, \cdot)\|^p_{L^\gamma(K)}\right] < \infty$$
for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,
4. the velocity field $\mathbf{u}$ is an $(\mathscr{F}_t)$-adapted random distribution and
$$\mathbb{E}\left[\int_0^T \|\mathbf{u}\|^2_{W^{1,2}(K)} \,\mathrm{d}t\right]^p < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

5. the momentum $\varrho\mathbf{u}$ satisfies $t \mapsto \langle \varrho\mathbf{u}, \boldsymbol{\varphi}\rangle \in C([0,T])$ for any $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho\mathbf{u}, \boldsymbol{\phi}\rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\varrho\mathbf{u}\|^p_{L^{\frac{2\gamma}{\gamma+1}}(K)}\right] < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

6. there exists $\mathscr{F}_0$-measurable random variables $(\varrho_0, \varrho_0\mathbf{u}_0) = (\varrho(0), \varrho\mathbf{u}(0))$ such that $\Lambda = \mathbb{P} \circ (\varrho_0, \varrho_0\mathbf{u}_0)^{-1}$;

7. for all $\psi \in C_c^\infty(\mathbb{R}^3)$ and $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$ and all $t \in [0,T]$, the following

$$\begin{aligned}
\langle \varrho(t), \psi\rangle &= \langle \varrho_0, \psi\rangle + \int_0^t \langle \varrho\mathbf{u}, \nabla\psi\rangle \mathrm{d}s, \\
\langle \varrho\mathbf{u}(t), \boldsymbol{\phi}\rangle &= \langle \varrho_0\mathbf{u}_0, \boldsymbol{\phi}\rangle + \int_0^t \langle \varrho\mathbf{u}\otimes\mathbf{u}, \nabla\boldsymbol{\phi}\rangle \mathrm{d}s - \nu\int_0^t \langle \nabla\mathbf{u}, \nabla\boldsymbol{\phi}\rangle \mathrm{d}s \\
&\quad - (\lambda+\nu)\int_0^t \langle \operatorname{div}\mathbf{u}, \operatorname{div}\boldsymbol{\phi}\rangle \mathrm{d}s + a\int_0^t \langle \varrho^\gamma, \operatorname{div}\boldsymbol{\phi}\rangle \mathrm{d}s \\
&\quad + \int_0^t \langle \Phi(\varrho, \varrho\mathbf{u})\mathrm{d}W\,,\, \boldsymbol{\phi}\rangle
\end{aligned} \tag{3.11}$$

hold $\mathbb{P}$-a.s.

Definition 3.2.6. If in addition to Definition 3.2.5,

1. the energy inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\overline{\varrho})\right](t)\,\mathrm{d}x + \nu\int_0^t\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + (\lambda+\nu)\int_0^t\int_{\mathbb{R}^3}|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s \leq \int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0,\overline{\varrho})\right]\mathrm{d}x \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\sum_{k\in\mathbb{N}}\varrho^{-1}|\mathbf{g}_k(x,\varrho,\varrho\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}s + M_R(t)
\end{aligned} \tag{3.12}$$

holds $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$ and $\overline{\varrho} > 0$ where

$$H(\varrho,\overline{\varrho}) = \frac{a}{(\gamma-1)}\left[\varrho^\gamma - \gamma\overline{\varrho}^{\gamma-1}(\varrho-\overline{\varrho}) - \overline{\varrho}^\gamma\right] \tag{3.13}$$

and M_R is a real-valued martingale starting at zero and given explicitly by

$$M_R(t) = \int_0^t \int_{\mathbb{R}^3} \mathbf{u} \cdot \Phi(\varrho, \varrho\mathbf{u}) \,\mathrm{d}x \,\mathrm{d}W; \tag{3.14}$$

2. and $(1.16)_1$ holds in the renormalized sense, i.e., for any $\phi \in C_c^\infty([0,T) \times \mathbb{R}^3)$ and $b \in C_b^1(\mathbb{R})$ such that $b'(z) = 0$ for all $z \geq M_b$, we have that

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^3} b(\varrho)\, \partial_t \phi \,\mathrm{d}x \,\mathrm{d}t = &\int_{\mathbb{R}^3} b\big(\varrho(0)\big)\, \phi(0) \,\mathrm{d}x \\ &+ \int_0^T \int_{\mathbb{R}^3} \big[b(\varrho)\mathbf{u}\big] \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}t \\ &- \int_0^T \int_{\mathbb{R}^3} \big[\big(b'(\varrho)\varrho - b(\varrho)\big)\mathrm{div}\mathbf{u}\big]\, \phi \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.15}$$

holds $\mathbb{P}$-a.s., then we call $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho, \mathbf{u}, W]$ a *finite energy weak martingale solution* of (1.16).

Remark 3.2.7. A similar notion as Definition 3.2.6 above also holds for functions defined on the periodic space $\mathbb{T}_L^3 = ([-L, L]|_{\{-L,L\}})^3 = (\mathbb{R}\,|\,2L\mathbb{Z})^3$ for any $L \geq 1$, rather than on the whole space $\mathbb{R}^3$. The precise formulation taken from [15, Definition 3.4.1] is given as follows for the relevant three dimensional case and for choice of $L = 1$:

Definition 3.2.8. Let $\Lambda = \Lambda(\varrho, \mathbf{m})$ be a Borel probability measure on $L^1(\mathbb{T}^3) \times L^1(\mathbb{T}^3)$ such that

$$\Lambda\{\varrho \geq 0\} = 1, \qquad \int_{L_x^1 \times L_x^1} \left| \int_{\mathbb{T}^3} \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \mathrm{d}x \right|^q \mathrm{d}\Lambda(\varrho, \mathbf{m}) < \infty \tag{3.16}$$

for any $q \geq 1$ and any function P of the form

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \,\mathrm{d}z. \tag{3.17}$$

Then $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho, \mathbf{u}, W]$ is a *dissipative martingale solution* of (1.16) with initial law Λ provided

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;

2. W is a $(\mathscr{F}_t)$-cylindrical Wiener process;

3. the density ϱ and the velocity $\mathbf{u}$ are $(\mathscr{F}_t)$-adapted random distributions and in addition, the density is non-negative $\mathbb{P}$-a.s.;

4. there exists $\mathscr{F}_0$-measurable random variables $(\varrho_0, \varrho_0\mathbf{u}_0) = (\varrho(0), \varrho\mathbf{u}(0))$ such that $\Lambda = \mathbb{P} \circ (\varrho(0), \varrho\mathbf{u}(0))^{-1}$,

5. for all $\psi \in C^\infty(\mathbb{T}^3)$ and $\boldsymbol{\phi} \in C^\infty(\mathbb{T}^3)$ and for all $\varphi \in C_c^\infty([0,T))$, the following

$$
\begin{aligned}
-\int_0^T \partial_t\varphi \int_{\mathbb{T}^3} \varrho\psi \,\mathrm{d}x\,\mathrm{d}t &= \varphi(0)\int_{\mathbb{T}^3} \varrho_0\psi\,\mathrm{d}x + \int_0^T \varphi \int_{\mathbb{T}^3} \varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t,\\
-\int_0^T \partial_t\varphi \int_{\mathbb{T}^3} \varrho\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t &= \varphi(0)\int_{\mathbb{T}^3} \varrho_0\mathbf{u}_0\cdot\boldsymbol{\phi}\,\mathrm{d}x\\
&\quad+\int_0^T \varphi\int_{\mathbb{T}^3}\big[\varrho\mathbf{u}\otimes\mathbf{u} - \nu\nabla\mathbf{u}\big] : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T \varphi\int_{\mathbb{T}^3}\big[a\varrho^\gamma - (\lambda+\nu)\mathrm{div}\,\mathbf{u}\big]\mathrm{div}\,\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sum_{k\in\mathbb{N}}\int_0^T\varphi\int_{\mathbb{T}^3}\mathbf{g}_k(\varrho,\varrho\mathbf{u})\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\beta_k
\end{aligned}
\tag{3.18}
$$

hold $\mathbb{P}$-a.s.;

6. the following energy inequality

$$
\begin{aligned}
&-\int_0^T \partial_t\varphi\int_{\mathbb{T}^3}\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\mathrm{d}x\,\mathrm{d}t + \nu\int_0^T\varphi\int_{\mathbb{T}^3}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+(\lambda+\nu)\int_0^T\varphi\int_{\mathbb{T}^3}|\mathrm{div}\,\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t \le \varphi(0)\int_{\mathbb{T}^3}\left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right]\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_0^T\varphi\int_{\mathbb{T}^3}\Big(\varrho^{-1}|\mathbf{g}_k(\varrho,\varrho\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}t + \mathbf{u}\cdot\mathbf{g}_k(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}\beta_k\Big)
\end{aligned}
\tag{3.19}
$$

holds $\mathbb{P}$-a.s. for all $\varphi \in C_c^\infty([0,T))$, $\varphi \ge 0$;

7. for any $\phi \in C^\infty(\mathbb{T}^3)$ and $b \in C^1(\mathbb{R})$ such that $b'(z) = 0$ for all $z \ge M_b$, we

have that

$$\begin{aligned} -\int_0^T \partial_t\varphi \int_{\mathbb{T}^3} b(\varrho)\,\phi \,\mathrm{d}x\,\mathrm{d}t &= \varphi(0)\int_{\mathbb{T}^3} b\big(\varrho_0\big)\,\phi\,\mathrm{d}x \\ &\quad + \int_0^T \varphi \int_{\mathbb{T}^3} \big[b(\varrho)\mathbf{u}\big]\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \varphi \int_{\mathbb{T}^3} \big[\big(b'(\varrho)\varrho - b(\varrho)\big)\mathrm{div}\mathbf{u}\big]\,\phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.20}$$

holds $\mathbb{P}$-a.s. for all $\varphi \in C_c^\infty([0,T))$.

Remark 3.2.9. Sometimes it is useful to have a different form of the energy estimate or energy inequality (3.12). With an abuse of notation, we shall also refer to

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho,\overline{\varrho})\right)(t)\mathrm{d}x\right]^p + \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^3}\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}s\right]^p \\ &\qquad \lesssim 1 + \mathbb{E}\left[\int_{\mathbb{R}^3}\left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H(\varrho_0,\overline{\varrho})\right)\,\mathrm{d}x\right]^p \end{aligned} \tag{3.21}$$

as a (moment) energy estimate. One instant when (3.21) is useful is in deriving uniform moment estimates for the purpose of compactness and its derivation from (3.12) is similar to the derivation of (3.41).

3.2.10 Formal derivation of the renormalized continuity equation

We present in this section, a formal derivation of (3.20) given the continuity equation $(1.16)_1$. For this, we consider a general smooth function $b(\varrho)$ belonging to the class $C^0[0,\infty)\cap C^1(0,\infty)$. By multiplying the continuity equation by $b'(\varrho)$, we gain

$$\begin{aligned} 0 &= b'(\varrho)\,\mathrm{d}\varrho + b'(\varrho)\,\mathrm{div}(\varrho\mathbf{u})\mathrm{d}t \\ &= \mathrm{d}\,[b(\varrho)] + \big[b'(\varrho)\,\varrho\,\mathrm{div}\,\mathbf{u} + b'(\varrho)\nabla\varrho\cdot\mathbf{u}\big]\mathrm{d}t. \end{aligned} \tag{3.22}$$

On the other hand,

$$b'(\varrho)\nabla\varrho\cdot\mathbf{u} = \nabla b(\varrho)\cdot\mathbf{u} = \mathrm{div}(b(\varrho)\mathbf{u}) - b(\varrho)\,\mathrm{div}\,\mathbf{u}. \tag{3.23}$$

Combining the two equations above gives

$$0 = \mathrm{d}\,[b(\varrho)] + \Big[b'(\varrho)\,\varrho\,\mathrm{div}\,\mathbf{u} + \mathrm{div}(b(\varrho)\mathbf{u}) - b(\varrho)\,\mathrm{div}\,\mathbf{u}\Big]\mathrm{d}t. \tag{3.24}$$

To finally gain (3.20), we proceed to integrate (3.24) against a test function.

3.2.11 Main Result

We finally state the main result in this chapter.

Theorem 3.2.12. *Let* $\overline{\varrho} > 0$, $\gamma > \frac{3}{2}$ *and assume that* Λ *is a Borel probability measure on* $L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)$ *satisfying*

$$\Lambda\Big\{(\varrho, \mathbf{m}) \in L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3) \,:\, 0 < M_1 \leq \varrho \leq M_2 \;\; a.e\Big\} = 1,$$

and

$$\int_{L^\gamma_x \times L^{\frac{2\gamma}{\gamma+1}}_x} \left| \int_{\mathbb{R}^3} \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + H(\varrho, \overline{\varrho}) \right) \mathrm{d}x \right|^p \mathrm{d}\Lambda(\varrho, \mathbf{m}) \leq c_p < \infty, \tag{3.25}$$

for all $0 \leq p < \infty$ *and constants* $M_1, M_2 > 0$. *Also assume that* (3.1)–(3.5) *holds. Then there exists a finite energy weak martingale solution of* (1.16) *in the sense of Definition 3.2.6 with initial law* Λ.

3.3 Uniform estimates and compactness arguments

We now start the proof of Theorem 3.2.12 by collecting uniform bounds on a family of approximations made up of solutions to periodic problems. We derive these bounds from an energy method and then proceed to showing compactness from tightness on the set on laws on these approximations.

3.3.1 Construction of the initial law

In order to obtain the initial law prescribed in item four of Definition 3.2.5, we consider the following family of cut-off functions

$$\begin{cases} \eta_L \in C_0^\infty\big([-L,L]^3\big), \qquad 0 \leq \eta_L \leq 1, \\ \eta_L \equiv 1 \text{ in } \big[-\frac{L}{2},\frac{L}{2}\big]^3 \end{cases} \tag{3.26}$$

defined for $L \geq 1$, and we let $\overline{\varrho} > 0$ be the anticipated far-field condition of density.

Given that Λ in Theorem 3.2.12 is a measure on a Polish space, we gain by Skorokhod's theorem (where our 'sequence' consists of just one pair of variables $(\varrho, \mathbf{m})$), the existence of some $\mathscr{F}_0$-measurable random variables $(\varrho_0,\, \mathbf{m}_0)$, $\varrho_0 > 0$ defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and having Λ as its law. We can further set $\mathbf{u}_0 := \frac{\mathbf{m}_0}{\varrho_0}$ to gain an additional variable $(\varrho_0,\, \mathbf{u}_0,\, \mathbf{m}_0)$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$.

Now with the construction of (3.26) above, we obtain the following family

$$\varrho_{0,L} = \eta_L \varrho_0 + (1-\eta_L)\overline{\varrho}, \quad \mathbf{m}_{0,L} := \varrho_{0,L}\mathbf{u}_{0,L} = \eta_L \frac{\sqrt{\varrho_{0,L}}}{\sqrt{\varrho_0}} \mathbf{m}_0 \tag{3.27}$$

of periodic functions (since η_L is periodic) having the property that for any $\omega \in \Omega$,

$$\varrho_{0,L}\big|_{\partial[-L,L]^3} = \overline{\varrho}, \quad \mathbf{m}_{0,L}\big|_{\partial[-L,L]^3} = 0 \tag{3.28}$$

and that

$$(\varrho_{0,L}, \mathbf{m}_{0,L}) \to (\varrho_0, \mathbf{m}_0) \quad \text{a.e. in} \quad \mathbb{R}^3 \tag{3.29}$$

as $L \to \infty$.

Now let $K \Subset \mathbb{R}^3$ be arbitrary and choose $L \gg 1$ such that $K \subset [-L,L]^3$. Then we have that $|\varrho_{0,L}| \lesssim \varrho_0^\gamma + \overline{\varrho}^\gamma$ and $|\mathbf{m}_{0,L}| \lesssim 1 + \mathbf{m}_0^{\frac{2\gamma}{\gamma+1}}$ holds uniformly in L. Furthermore, by the assumption on the law in Theorem 3.2.12, it follows that $\varrho_0^\gamma + \overline{\varrho}^\gamma \in L^1(K)$

and $1+\mathbf{m}_0^{\frac{2\gamma}{\gamma+1}} \in L^1(K)$. Hence by dominated convergence,

$$(\varrho_{0,L}, \mathbf{m}_{0,L}) \to (\varrho_0, \mathbf{m}_0) \quad \text{in} \quad L^\gamma(K) \times L^{\frac{2\gamma}{\gamma+1}}(K) \tag{3.30}$$

a.s. Subsequently, we gain

$$\Lambda_L = \mathbb{P} \circ (\varrho_{0,L}, \mathbf{m}_{0,L})^{-1} \overset{*}{\rightharpoonup} \mathbb{P} \circ (\varrho_0, \mathbf{m}_0)^{-1} = \Lambda \tag{3.31}$$

in the sense of measures on $L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)$ by the arbitrariness of $K \Subset \mathbb{R}^3$.

3.3.2 A priori bounds

By periodicity and invariance, we gain from [15, Theorem 4.0.2], the existence of a family of dissipative martingale solutions

$$\left[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho_L, \mathbf{u}_L, W\right] \tag{3.32}$$

in the sense of Definition 3.2.8 which are defined for $\overline{\mathrm{d}\mathbb{P}\otimes\mathrm{d}t}$ a.e. $(\omega, t) \in \Omega \times [0,T]$ on the periodic domains $\mathbb{T}^3_L$ for $L \geq 1$. Without loss of generality, we have chosen the family of solutions (3.32) to be defined on the same stochastic basis as well as driven by the same Wiener process. Furthermore, we choose $L \gg 1$ large enough so that for the compact set K in (3.1), we have that $K \subset \mathbb{T}^3_L$.

The prescribed laws for the solutions (3.32) are the Borel probability measures $\Lambda_L = \mathbb{P} \circ (\varrho_{0,L}\,,\, \varrho_{0,L}\mathbf{u}_{0,L})^{-1}$ defined on $L^1(\mathbb{T}^3_L) \times L^1(\mathbb{T}^3_L)$ (see Section 3.3.1 for the construction of the corresponding initial random variables) and which are assumed to satisfy

$$\Lambda_L\{\varrho \geq 0\} = 1, \qquad \int_{L^1_x \times L^1_x} \left| \int_{\mathbb{T}^3_L} \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \mathrm{d}x \right|^q \mathrm{d}\Lambda_L(\varrho, \mathbf{m}) < \infty, \tag{3.33}$$

for some $q \geq 4$ with

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma}.$$

We will justify the boundedness of the initial data shortly, see (3.38) below.

Now, in analogy with (3.4)–(3.5), the corresponding diffusion coefficients $\Phi(\varrho_L, \mathbf{m}_L) : \mathfrak{U} \to L^1(\mathbb{T}^3_L)$ are defined by

$$\Phi(\varrho_L, \mathbf{m}_L)e_k = \mathbf{g}_k\big(\cdot, \varrho_L(\cdot), \mathbf{m}_L(\cdot)\big)$$

with the following C^1- functions $\mathbf{g}_k = \mathbf{g}_k\big(x, \varrho_L, \mathbf{m}_L\big) : \mathbb{T}^3_L \times \mathbb{R}^3_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the bounds

$$\begin{aligned} \big|\mathbf{g}_k\big(x, \varrho_L, \mathbf{m}_L\big)\big| &\leq c_k\big(\varrho_L + \big|\mathbf{m}_L\big|\big), \\ \big|\nabla_{\varrho, \mathbf{m}}\, \mathbf{g}_k\big(x, \varrho_L, \mathbf{m}_L\big)\big| &\leq c_k \end{aligned} \tag{3.34}$$

for squared-summable constant $(c_k)_{k\in\mathbb{N}} \subset [0, \infty)$ which are bounded uniformly in $x \in \mathbb{T}^3_L$, i.e., $\sum_{k\in\mathbb{N}} c_k^2 \lesssim 1$.

We can therefore conclude from (3.19) that for all $\varphi \in C^\infty_c([0,T))$, $\varphi \geq 0$, the following inequality

$$\begin{aligned} \int_0^T \varphi \int_{\mathbb{T}^3_L} \mathbb{S}(\nabla \mathbf{u}_L) : \nabla \mathbf{u}_L \, \mathrm{d}x\mathrm{d}t \leq \varphi(0) \int_{\mathbb{T}^3_L} \left[\frac{1}{2}\varrho_{0,L}|\mathbf{u}_{0,L}|^2 + P(\varrho_{0,L})\right] \mathrm{d}x \\ + \int_0^T \partial_t \varphi \int_{\mathbb{T}^3_L} \left[\frac{\varrho_L|\mathbf{u}_L|^2}{2} + P(\varrho_L)\right](t) \, \mathrm{d}x \, \mathrm{d}t \\ + \sum_{k\in\mathbb{N}} \int_0^T \varphi \int_{\mathbb{T}^3_L} \mathbf{u}_L \cdot \mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L) \mathrm{d}x \, \mathrm{d}\beta_k \\ + \int_0^T \varphi \int_{\mathbb{T}^3_L} \sum_{k\in\mathbb{N}} \frac{|\mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L)|^2}{2\varrho_L} \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.35}$$

holds $\mathbb{P}$-a.s. and so if we use a sequence of non-negative compactly supported smooth functions φ_m to approximate $t \mapsto \chi_{[0,t]}$, it follows that

$$\begin{aligned}\int_0^t \int_{\mathbb{T}_L^3} \mathbb{S}(\nabla \mathbf{u}_L) : \nabla \mathbf{u}_L \,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{T}_L^3} \left[\frac{\varrho_L |\mathbf{u}_L|^2}{2} + P(\varrho_L) \right](t)\,\mathrm{d}x \\ \leq \int_{\mathbb{T}_L^3} \left[\frac{1}{2} \varrho_{0,L} |\mathbf{u}_{0,L}|^2 + P(\varrho_{0,L}) \right] \mathrm{d}x \\ + \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}_L^3} \mathbf{u}_L \cdot \mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L) \mathrm{d}x\,\mathrm{d}\beta_k \\ + \int_0^t \int_{\mathbb{T}_L^3} \sum_{k\in\mathbb{N}} \frac{|\mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L)|^2}{2\varrho_L} \,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{3.36}$$

holds $\mathbb{P}$-a.s. for a.a. $t \in [0, T]$.

Now we note that the energy inequality (3.36) is preserved under any affine perturbation of $P(\varrho_L)$. So for the anticipated steady state $\overline{\varrho}$, the choice of perturbation function

$$\frac{-a}{\gamma - 1} \left[\overline{\varrho}^\gamma - \gamma \overline{\varrho}^{\gamma-1} (\varrho_L - \overline{\varrho}) \right]$$

leading to

$$H\big(\varrho_L, \overline{\varrho}\big) = P(\varrho_L) - P'(\overline{\varrho})\big(\varrho_L - \overline{\varrho}\big) - P(\overline{\varrho})$$

preserves the resulting augmented energy inequality with the help of mass conservation. i.e.,

$$\begin{aligned}\int_{\mathbb{T}_L^3} \left[\frac{\varrho_L |\mathbf{u}_L|^2}{2} + H\big(\varrho_L, \overline{\varrho}\big) \right](t)\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}_L^3} \mathbb{S}(\nabla \mathbf{u}_L) : \nabla \mathbf{u}_L \,\mathrm{d}x\,\mathrm{d}s \\ \leq \int_{\mathbb{T}_L^3} \left[\frac{1}{2} \varrho_{0,L} |\mathbf{u}_{0,L}|^2 + H\big(\varrho_{0,L}, \overline{\varrho}\big) \right] \mathrm{d}x + \int_0^t \int_{\mathbb{T}_L^3} \mathbf{u}_L \cdot \Phi(\varrho_L, \varrho_L \mathbf{u}_L) \mathrm{d}x\,\mathrm{d}W_L \\ + \int_0^t \int_{\mathbb{T}_L^3} \sum_{k\in\mathbb{N}} \frac{1}{2} \frac{|\mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L)|^2}{\varrho_L} \,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{3.37}$$

holds $\mathbb{P}$-a.s. for any $t \in [0, T]$.

However, by a straightforward computation, one can verify that for a.e. $(\omega, x) \in \Omega \times \mathbb{T}_L^3$, the structure of the data sequence as presented in Section 3.3.1 yields

$\varrho_{0,L}|\mathbf{u}_{0,L}|^2 \leq \varrho_0|\mathbf{u}_0|^2$ and by convexity,

$$H\big(\varrho_{0,L}\,,\,\overline{\varrho}\big) \leq \eta_L H\big(\varrho_0\,,\,\overline{\varrho}\big) + (1-\eta_L)H\big(\overline{\varrho}\,,\,\overline{\varrho}\big) \leq H\big(\varrho_0\,,\,\overline{\varrho}\big).$$

As such, without even having to pass to the limit $L \to \infty$, we gain from the initial energy on the right-hand side of (3.37) that the inequality

$$\int_{\mathbb{T}^3_L} \left[\frac{1}{2}\varrho_{0,L}|\mathbf{u}_{0,L}|^2 + H\big(\varrho_{0,L},\overline{\varrho}\big)\right] \mathrm{d}x \leq \int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H\big(\varrho_0,\overline{\varrho}\big)\right] \mathrm{d}x \tag{3.38}$$

holds $\mathbb{P}$-a.s.

Now due to (3.34), we have that for any $1 \leq p < \infty$,

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \int_{\mathbb{T}^3_L} \sum_{k\in\mathbb{N}} \frac{|\mathbf{g}_k(\varrho_L,\varrho_L\mathbf{u}_L)|^2}{2\varrho_L} \,\mathrm{d}x\,\mathrm{d}s \right|^p \\
&\leq \mathbb{E}\left(\int_0^T \int_{\mathbb{T}^3_L} \sum_{k\in\mathbb{N}} \frac{|\mathbf{g}_k(\varrho_L,\varrho_L\mathbf{u}_L)|^2}{2\varrho_L} \,\mathrm{d}x\,\mathrm{d}s \right)^p \\
&\lesssim \mathbb{E}\left(\int_0^T \int_K \varrho_L^{-1}\big(\varrho_L^2 + |\varrho_L\mathbf{u}_L|^2\big) \,\mathrm{d}x\,\mathrm{d}s \right)^p \\
&\lesssim_p \mathbb{E}\int_0^T \left(\int_K \big(1+\varrho_L^\gamma + \varrho_L|\mathbf{u}_L|^2\big)\,\mathrm{d}x \right)^p \mathrm{d}s
\end{aligned} \tag{3.39}$$

for $K \Subset \mathbb{R}^3$ in (3.1) and where all constants are independent of L. We have also used the inequality $\varrho_L \leq 1 + \varrho_L^\gamma$.

Also, by the use of the Burkholder–Davis–Gundy inequality, Hölder inequality and

Young's inequality, we gain using (3.34),

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\Big|\int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{T}^3_L} \mathbf{u}_L\cdot \mathbf{g}_k(\varrho_L,\varrho_L\mathbf{u}_L)\mathrm{d}x\,\mathrm{d}\beta_k\Big|\right]^p \\
&\lesssim \mathbb{E}\left[\int_0^T \sum_{k\in\mathbb{N}}\left(\int_{\mathbb{T}^3_L} \mathbf{u}_L\cdot \mathbf{g}_k(\varrho_L,\varrho_L\mathbf{u}_L)\mathrm{d}x\right)^2\mathrm{d}s\right]^{\frac{p}{2}} \\
&\lesssim \mathbb{E}\left[\int_0^T \sum_{k\in\mathbb{N}}\left(\int_{\mathbb{T}^3_L} |\sqrt{\varrho_L}\mathbf{u}_L|^2\mathrm{d}x\right)\left(\int_{\mathbb{T}^3_L} \left|\frac{\mathbf{g}_k(\varrho_L,\varrho_L\mathbf{u}_L)}{\sqrt{\varrho_L}}\right|^2\mathrm{d}x\right)\mathrm{d}s\right]^{\frac{p}{2}} \\
&\leq \epsilon\,\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^3_L}|\sqrt{\varrho_L}\mathbf{u}_L|^2\mathrm{d}x\right)^p \\
&\quad + c(\epsilon)\,\mathbb{E}\int_0^T\left(\int_K (1+\varrho_L^\gamma+\varrho_L|\mathbf{u}_L|^2)\mathrm{d}x\right)^p\mathrm{d}s
\end{aligned}
\tag{3.40}
$$

for an arbitrarily small $\epsilon>0$.

By taking the pth-moment of the supremum in (3.37)–(3.38) and applying Gronwall's lemma, we obtain by using (3.38)–(3.40), the inequality

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{T}^3_L}\left(\frac{\varrho_L|\mathbf{u}_L|^2}{2}+H\big(\varrho_L,\overline{\varrho}\big)\right)\mathrm{d}x\right]^p \\
&\qquad + \mathbb{E}\left[\int_0^T\int_{\mathbb{T}^3_L}\mathbb{S}(\nabla\mathbf{u}_L):\nabla\mathbf{u}_L\,\mathrm{d}x\,\mathrm{d}s\right]^p \\
&\qquad \lesssim_{p,\mathrm{vol}(K)} \left(1+\mathbb{E}\left[\int_{\mathbb{T}^3_L}\left[\frac{1}{2}\varrho_{0,L}|\mathbf{u}_{0,L}|^2+H(\varrho_{0,L},\overline{\varrho})\right]\mathrm{d}x\right]^p\right) \\
&\qquad \lesssim_{p,\mathrm{vol}(K)} \left(1+\mathbb{E}\left[\int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+H(\varrho_0,\overline{\varrho})\right]\mathrm{d}x\right]^p\right)
\end{aligned}
\tag{3.41}
$$

where $c_{p,\mathrm{vol}(K)}$ is in particular, independent of L. Now by (3.25), the last term in (3.41) is finite since

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+H(\varrho_0,\overline{\varrho})\right]\mathrm{d}x\right]^p \\
&= \int_{L^\gamma_x\times L^{\frac{2\gamma}{\gamma+1}}_x}\left|\int_{\mathbb{R}^3}\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+H(\varrho,\overline{\varrho})\right)\mathrm{d}x\right|^p\mathrm{d}\Lambda(\varrho,\mathbf{m})\lesssim 1.
\end{aligned}
\tag{3.42}
$$

By using the estimate $a^p + b^p \lesssim (a+b)^p$, we can combine (3.38) and (3.42) so that

$$\begin{aligned} \varrho_{0,L}^{\gamma} &\in L^p\big(\Omega; L^1(\mathbb{T}_L^3)\big), \\ \varrho_{0,L}\big|\mathbf{u}_{0,L}\big|^2 &\in L^p\big(\Omega; L^1(\mathbb{T}_L^3)\big) \end{aligned} \tag{3.43}$$

uniformly in L for any $1 \leq p < \infty$ or that

$$\begin{aligned} \varrho_{0,L} &\in L^q\big(\Omega; L^{\gamma}(\mathbb{T}_L^3)\big), \\ \sqrt{\varrho_{0,L}}\mathbf{u}_{0,L} &\in L^q\big(\Omega; L^2(\mathbb{T}_L^3)\big) \end{aligned} \tag{3.44}$$

uniformly in L for any $2 \leq q < \infty$.

For $d\mathbb{P}$ a.e. $\omega \in \Omega$, we gain by applying Hölder's inequality that

$$\begin{aligned} \big\|\mathbf{m}_{0,L}\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}_L^3)} &= \big\|\sqrt{\varrho}_{0,L}\sqrt{\varrho}_{0,L}\mathbf{u}_{0,L}\big\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}_L^3)} \\ &\leq \big\|\sqrt{\varrho}_{0,L}\big\|_{L^{2\gamma}(\mathbb{T}_L^3)} \big\|\sqrt{\varrho}_{0,L}\mathbf{u}_{0,L}\big\|_{L^2(\mathbb{T}_L^3)} \\ &= \Big(\big\|\varrho_{0,L}\big\|_{L^{\gamma}(\mathbb{T}_L^3)} \big\|\varrho_{0,L}|\mathbf{u}_{0,L}|^2\big\|_{L^1(\mathbb{T}_L^3)}\Big)^{\frac{1}{2}} \end{aligned} \tag{3.45}$$

where $\mathbf{m}_{0,L} := \varrho_{0,L}\mathbf{u}_{0,L}$. As such, we further gain from (3.43)–(3.44) and Young's inequality,

$$\mathbb{E}\,\big\|\mathbf{m}_{0,L}\big\|^{2p}_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}_L^3)} \lesssim_p \mathbb{E}\,\big\|\varrho_{0,L}\big\|^{2p}_{L^{\gamma}(\mathbb{T}_L^3)} + \mathbb{E}\,\big\|\varrho_{0,L}|\mathbf{u}_{0,L}|^2\big\|^{2p}_{L^2(\mathbb{T}_L^3)} < \infty \tag{3.46}$$

uniformly in L for any $1 \leq p < \infty$. Thus,

$$\mathbf{m}_{0,L} \in L^q\big(\Omega; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}_L^3)\big) \tag{3.47}$$

uniformly in L for any $2 \leq q < \infty$.

Furthermore, we obtain from (3.41)–(3.42), the following uniform bounds in L

$$\begin{aligned} &\mathbb{E}\left| \sup_{t\in[0,T]} \left\| \varrho_L |\mathbf{u}_L|^2 \right\|_{L^1(\mathbb{T}^3_L)} \right|^p \lesssim 1, \\ &\mathbb{E}\left| \left(\int_0^T \left\| \nabla \mathbf{u}_L \right\|^2_{L^2(\mathbb{T}^3_L)} \mathrm{d}t \right)^{\frac{1}{2}} \right|^p \lesssim 1, \\ &\mathbb{E}\left| \sup_{t\in[0,T]} \left\| P(\varrho_L) \right\|_{L^1(\mathbb{T}^3_L)} \right|^p \lesssim 1. \end{aligned} \tag{3.48}$$

Note that the estimates in (3.48) are global but unfortunately, do not include all necessary quantities. In the following, we derive local estimates with respect to balls B_r which will depend on their radius $r > 0$.

Firstly, a consequence of $(3.48)_3$ is

$$\mathbb{E}\left| \sup_{t\in[0,T]} \left\| \varrho_L \right\|_{L^\gamma(B_r)} \right|^p \lesssim 1 \tag{3.49}$$

uniformly in L (but depending on r) for an isentropic pressure $p(z) = az^\gamma$ in (3.17). If $B_r \subset \mathbb{T}^3_L$, this follows in an obvious way from the definition of the pressure potential P. Otherwise we cover $B_r \subset \mathbb{R}^3$ by tori to which ϱ_L is extended by means of periodicity. The number of necessary tori depends on r but is independent of L. To see this, we notice that since $\mathrm{vol}(B_r) \approx c(\pi)r^3$ and $\mathrm{vol}(\mathbb{T}^3_L) \approx c(\pi)L^3$, we will require $\mathcal{O}\big(\frac{r^3}{L^3}\big)$ number of tori to cover B_r. But since $L \geq 1$, we infact require $\mathcal{O}(r^3)$ (which is independent of L) number of such tori to cover B_r.

We also observe that none of the bounds in (3.48) directly controls the amplitude of $\mathbf{u}_L$ uniformly in L. However using the conservation of mass, the Sobolev-Poincaré inequality and the condition that $\gamma > \frac{3}{2}$, the following holds

$$\begin{aligned} \|\varrho_0\|_{L^1(B_r)} \left|(\mathbf{u}_L)_{B_r}\right| &= \left| \int_{B_r} \varrho \,(\mathbf{u}_L)_{B_r} \mathrm{d}x \right| \\ &\lesssim \int_{B_r} \varrho \left|(\mathbf{u}_L)_{B_r} - \mathbf{u}_L\right| \mathrm{d}x + \int_{B_r} \varrho_L |\mathbf{u}_L| \mathrm{d}x \\ &\lesssim \|\varrho_L\|_{L^\gamma(B_r)} \left\|(\mathbf{u}_L)_{B_r} - \mathbf{u}_L\right\|_{L^{\gamma'}(B_r)} + \|\sqrt{\varrho_L}\|_{L^2(B_r)} \|\sqrt{\varrho_L}\mathbf{u}_L\|_{L^2(B_r)} \\ &\lesssim_r \|\varrho_L\|_{L^\gamma(B_r)} \left\|(\mathbf{u}_L)_{B_r} - \mathbf{u}_L\right\|_{L^6(B_r)} + \|\sqrt{\varrho_L}\|_{L^{2\gamma}(B_r)} \|\sqrt{\varrho_L}\mathbf{u}_L\|_{L^2(B_r)} \\ &\lesssim_r \|\varrho_L\|_{L^\gamma(B_r)} \|\nabla \mathbf{u}_L\|_{L^2(B_r)} + \|\varrho_L\|_{L^\gamma(B_r)} + \left\|\varrho_L |\mathbf{u}_L|^2\right\|_{L^1(B_r)}, \end{aligned}$$

where $(\mathbf{u}_L)_{B_r}$ is the average of $\mathbf{u}_L$ over the ball B_r. We have also used the fact that $\gamma' = \gamma/(\gamma-1) < 6$ and applied Young's inequality. Consequently,

$$\begin{aligned}\|\varrho_0\|^2_{L^1(B_r)} \int_0^\tau |(\mathbf{u}_L)_{B_r}|^2 \,\mathrm{d}t \leq\, & c(r) \sup_{t\in[0,\tau]} \|\varrho_L\|^2_{L^\gamma(B_r)} \int_0^\tau \|\nabla \mathbf{u}_L\|^2_{L^2(B_r)}\mathrm{d}t \\ & + c\tau \sup_{t\in(0,\tau)} \Big(\|\varrho_L\|^2_{L^\gamma(B_r)} + \big\|\varrho_L|\mathbf{u}_L|^2\big\|^2_{L^1(B_r)}\Big).\end{aligned} \tag{3.50}$$

In view of the bounds established in (3.48)–(3.49) and the assumptions on the initial law, we can conclude that

$$\mathbf{u}_L \in L^p(\Omega; L^2(0,T; W^{1,2}(B_r))). \tag{3.51}$$

uniformly in L.

Now for $r > 0$, we can use Hölder's inequality and Young's inequality to get for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$ a.e. $(\omega,t)\in\Omega\times[0,T]$,

$$\begin{aligned}\|\varrho_L\mathbf{u}_L\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)} &\leq \|\sqrt{\varrho_L}\|_{L^{2\gamma}(B_r)}\|\sqrt{\varrho_L}\mathbf{u}_L\|_{L^2(B_r)} \\ &= \|\varrho_L\|^{\frac{1}{2}}_{L^\gamma(B_r)}\|\sqrt{\varrho_L}\mathbf{u}_L\|_{L^2(B_r)} \\ &\lesssim \|\varrho_L\|_{L^\gamma(B_r)} + \big\|\varrho_L|\mathbf{u}_L|^2\big\|_{L^1(B_r)}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\left|\sup_{t\in[0,T]}\|\varrho_L\mathbf{u}_L\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p \lesssim_p\, & \mathbb{E}\left|\sup_{t\in[0,T]}\|\varrho_L\|_{L^\gamma(B_r)}\right|^p \\ & + \mathbb{E}\left|\sup_{t\in[0,T]}\big\|\varrho_L|\mathbf{u}_L|^2\big\|_{L^1(B_r)}\right|^p.\end{aligned} \tag{3.52}$$

Also, we can use the continuous embedding $W^{1,2}(B_r)\hookrightarrow L^6(B_r)$ and Hölder's inequality to get for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$ a.e. $(\omega,t)\in\Omega\times[0,T]$,

$$\begin{aligned}\|\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L\|_{L^{\frac{6\gamma}{4\gamma+3}}(B_r)} &\leq \|\varrho_L\mathbf{u}_L\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\|\mathbf{u}_L\|_{L^6(B_r)} \\ &\lesssim_r \|\varrho_L\mathbf{u}_L\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\|\mathbf{u}_L\|_{W^{1,2}(B_r)}.\end{aligned}$$

As such by Hölder inequality in time, we gain

$$\begin{aligned}\mathbb{E}\left|\left(\int_0^T \left\|\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L\right\|^2_{L^{\frac{6\gamma}{4\gamma+3}}(B_r)}\,\mathrm{d}t\right)^{\frac{1}{2}}\right|^p \lesssim_r \mathbb{E}\left|\sup_{t\in[0,T]}\left\|\varrho_L\mathbf{u}_L\right\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p\\ \times\mathbb{E}\left|\left(\int_0^T\left\|\mathbf{u}_L\right\|^2_{W^{1,2}(B_r)}\,\mathrm{d}t\right)^{\frac{1}{2}}\right|^p.\end{aligned} \tag{3.53}$$

Given $(3.48)_1$, (3.49) and (3.51), we may therefore conclude from (3.52) and (3.53) that

$$\begin{aligned}\mathbb{E}\left|\sup_{t\in[0,T]}\left\|\varrho_L\mathbf{u}_L\right\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p\lesssim_r 1,\\ \mathbb{E}\left|\left(\int_0^T\left\|\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L\right\|^2_{L^{\frac{6\gamma}{4\gamma+3}}(B_r)}\,\mathrm{d}t\right)^{\frac{1}{2}}\right|^p\lesssim_r 1,\end{aligned} \tag{3.54}$$

uniformly in L for $r>0$.

3.3.3 Pressure estimate

To avoid the situation where the limiting pressure is just a measure, it is important to improve the regularity of density. This required improvement is given in the following lemma.

Lemma 3.3.4. Let $B_r\subset\mathbb{R}^3$ be a ball of radius $r>0$. Then for all $\Theta<\frac{2}{3}\gamma-1$, we have that

$$\mathbb{E}\int_0^T\int_{B_r}\varrho_L^{\gamma+\Theta}\,\mathrm{d}x\,\mathrm{d}t\lesssim_r 1 \tag{3.55}$$

where the constant is independent of L large enough.

Proof. We let B be an arbitrary ball and set $\mathcal{B}:=\nabla\Delta_B^{-1}$ where Δ_B^{-1} is the fundamental solution of the Laplacian $-\Delta$ having homogeneous Dirichlet boundary condition. Then for for all $p\in(1,\infty)$, the operator

$$\mathcal{B}:L^p(B)\to W^{1,p}(B),\quad \mathcal{B}:W^{-1,p}(B)\to L^p(B),$$

is such that

$$\mathrm{div}\mathcal{B}(f) = f, \quad f \in L^p(B), \tag{3.56}$$

$$\|\mathcal{B}(f)\|_{W^{1,p}(B)} \lesssim_{p,B} \|f\|_{L^p(B)}, \quad f \in L^p(B). \tag{3.57}$$

Also, it follows from Sobolev's inequality and (3.57) that

$$\big\|\mathcal{B}(f)\big\|_{L^{\frac{3q}{3-q}}(B)} \leq \big\|\mathcal{B}\,(f)\,\big\|_{W^{1,q}(B)} \lesssim_{q,B} \big\|f\big\|_{L^q(B)} \tag{3.58}$$

holds for $1 \leq q < 3$ and $f \in L^q(B)$. Also, for $r > 3$ and $f \in L^r(B)$,

$$\big\|\mathcal{B}(f)\big\|_{L^\infty(B)} \leq \big\|\mathcal{B}\,(f)\,\big\|_{W^{1,r}(B)} \lesssim_q \big\|f\big\|_{L^r(B)}. \tag{3.59}$$

Now we note that for $L \gg 1$ large enough, $B_r \cap \mathbb{T}^3_L = B_r$ for $r > 0$ fixed. On this ball, we intend to use the localized version of the standard procedure where we test the momentum equation with $\eta\mathcal{B}(\varrho^\Theta) = \eta\nabla\Delta^{-1}_{\tilde{B}_r}(\varrho^\Theta)$ where $\eta \in C^\infty_0(\tilde{B}_r)$ with $\eta = 1$ in B_r and $B_r \Subset \tilde{B}_r$. To do this however, we first replace the map $\varrho \mapsto \varrho^\Theta$ with the function $b \in C^1_c(\mathbb{R})$ and apply Itô's formula to the function $f(b, \mathbf{m}) = \int_{\mathbb{R}^3} \eta\mathbf{m} \cdot \mathcal{B}(b(\varrho))\,\mathrm{d}x$. Since f is linear in $\mathbf{m}$, no second-order derivative in this component appears. Also, the quadratic variance of $b(\varrho)$ is zero since the renormalized continuity equation is deterministic.

Now notice that the operator $\mathcal{B}$ commutes with the time derivative (but not with the spatial derivative). Furthermore, to improve this weak formulation of the renormalized continuity equation into an analytically strong statement, we consider the usual mollifier $\wp_k$ and define its convolution with a locally integrable function f as $[f]_k := f * \wp_k$. Then we gain from renormalized continuity equation that

$$\begin{aligned}\eta\mathcal{B}\big[b(\varrho_L(t))\big]_\kappa = \eta\mathcal{B}\big[b(\varrho_{0,L})\big]_\kappa - \int_0^t \eta\mathcal{B}\,\mathrm{div}\big[b(\varrho_L)\mathbf{u}_L\big]_\kappa \mathrm{d}s \\ - \int_0^t \eta\mathcal{B}\big[\big(b'(\varrho_L)\,\varrho_L - b(\varrho_L)\big)\mathrm{div}\,\mathbf{u}_L\big]_\kappa \mathrm{d}s\end{aligned} \tag{3.60}$$

is satisfied $\mathbb{P}$-a.s. for any $t \in [0, T]$ and a.e. $x \in \mathbb{R}^3$.

Also, since (3.32) satisfies the second weak formulation in (3.11), a similar smoothing argument as in (3.60) yields

$$
\begin{aligned}
\big[(\varrho_L \mathbf{u}_L)(t)\big]_\kappa &= \big[\varrho_{0,L}\mathbf{u}_{0,L}\big]_\kappa - \int_0^t \operatorname{div}\big[\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L\big]_\kappa \mathrm{d}s - a\int_0^t \nabla\big[\varrho_L^\gamma\big]_\kappa \mathrm{d}s \\
&+ \nu \int_0^t \Delta\big[\mathbf{u}_L\big]_\kappa \mathrm{d}s + (\lambda+\nu)\int_0^t \nabla\operatorname{div}\big[\mathbf{u}_L\big]_\kappa \mathrm{d}s + \int_0^t \big[\Phi(\varrho_L\, \varrho_L \mathbf{u}_L)\big]_\kappa \mathrm{d}W
\end{aligned}
\tag{3.61}
$$

$\mathbb{P}$-a.s. for any $t \in [0,T]$ and a.e. $x \in \mathbb{R}^3$.

By applying the classical Itô product rule for continuous martingales, we gain from (3.60) and (3.61),

$$
\begin{aligned}
\mathbb{E}\int_{\mathbb{R}^3} \eta\big[\varrho_L\mathbf{u}_L\big]_\kappa \cdot \mathcal{B}\big[b(\varrho_L)\big]_\kappa \,\mathrm{d}x &= \mathbb{E}\int_{\mathbb{R}^3} \eta\big[\varrho_{0,L}\mathbf{u}_{0,L}\big]_\kappa \cdot \mathcal{B}\big[b(\varrho_{0,L})\big]_\kappa \,\mathrm{d}x \\
&- \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big[\varrho_L\mathbf{u}_L\big]_\kappa \cdot \mathcal{B}\operatorname{div}\big[b(\varrho_L)\,\mathbf{u}_L\big]_\kappa \mathrm{d}x\,\mathrm{d}s \\
&- \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big[\varrho_L\mathbf{u}_L\big]_\kappa \cdot \mathcal{B}\big[\varrho_L b'(\varrho_L)\operatorname{div}\mathbf{u}_L\big]_\kappa \mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big[\varrho_L\mathbf{u}_L\big]_\kappa \cdot \mathcal{B}\big[b(\varrho_L)\operatorname{div}\mathbf{u}_L\big]_\kappa \mathrm{d}x\,\mathrm{d}s \\
&- \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\operatorname{div}\big[\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L\big]_\kappa \cdot \mathcal{B}\big[b(\varrho_L)\big]_\kappa \,\mathrm{d}x\mathrm{d}s \\
&+ \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\nu\Delta[\mathbf{u}_L]_\kappa \cdot \mathcal{B}\big[b(\varrho_L)\big]_\kappa \,\mathrm{d}x\mathrm{d}s \\
&+ \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta(\lambda+\nu)\mathcal{B}\big[b(\varrho_L)\big]_\kappa \cdot \nabla\operatorname{div}[\mathbf{u}_L]_\kappa \,\mathrm{d}x\,\mathrm{d}s \\
&- \mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\, a\nabla[\varrho_L^\gamma]_\kappa \cdot \mathcal{B}\big[b(\varrho_L)\big]_\kappa \,\mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_{\mathbb{R}^3}\int_0^t \eta\mathcal{B}\big[b(\varrho_L)\big]_\kappa \big[\Phi(\varrho_L, \varrho_L\mathbf{u}_L)\big]_\kappa \,\mathrm{d}W\,\mathrm{d}x.
\end{aligned}
\tag{3.62}
$$

Remark 3.3.5. Notice that without smoothening, a term such as

$$
\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\nu\Delta\mathbf{u}_L \cdot \mathcal{B}\, b(\varrho_L)\,\mathrm{d}x\mathrm{d}s
$$

will not make sense by virtue of the weak spatial regularity (3.51) for velocity.

Now based on the various a priori bounds derived in Section 3.3.2, we can integrate by parts, nonsensical terms of (3.62) in the spirit of Remark 3.3.5. Then by the usual properties of mollifiers, we are able to pass to the limit $\kappa \to 0$ to gain

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^3} \eta\big(\varrho_L \mathbf{u}_L\big) \cdot \mathcal{B}\left(b(\varrho_L)\right) \,\mathrm{d}x = \mathbb{E}\int_{\mathbb{R}^3} \eta\big(\varrho_{0,L}\mathbf{u}_{0,L}\big) \cdot \mathcal{B}\left[b(\varrho_{0,L})\right] \,\mathrm{d}x \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta\big(\varrho_L \mathbf{u}_L\big) \cdot \mathcal{B}\left[\mathrm{div}(b(\varrho_L)\,\mathbf{u}_L)\right] \mathrm{d}x\,\mathrm{d}s \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta\big(\varrho_L \mathbf{u}_L\big) \cdot \mathcal{B}\left[\varrho_L b'(\varrho_L)\,\mathrm{div}\,\mathbf{u}_L\right] \mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta\big(\varrho_L \mathbf{u}_L\big) \cdot \mathcal{B}\left[b(\varrho_L)\mathrm{div}\,\mathbf{u}_L\right] \mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta(\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L) : \nabla\mathcal{B}\big(b(\varrho_L)\big)\,\mathrm{d}x\mathrm{d}s \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta\nu\nabla\mathbf{u}_L : \nabla\mathcal{B}\big(b(\varrho_L)\big)\,\mathrm{d}x\mathrm{d}s \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta(\lambda+\nu) b(\varrho_L)\,\mathrm{div}\mathbf{u}_L\,\mathrm{d}x\,\mathrm{d}s \qquad (3.63)\\
&+ \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta a\varrho_L^\gamma\, b(\varrho_L)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_{\mathbb{R}^3} \int_0^t \eta\mathcal{B}\big(b(\varrho_L)\big)\Phi(\varrho_L, \varrho_L\mathbf{u}_L)\,\mathrm{d}W\,\mathrm{d}x \\
&+ \mathbb{E}\int_0^t \int_{\mathbb{R}^3} (\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L) : \nabla\eta \otimes \mathcal{B}\big(b(\varrho_L)\big)\,\mathrm{d}x\mathrm{d}s \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} \nu\nabla\mathbf{u}_L : \nabla\eta \otimes \mathcal{B}\big(b(\varrho_L)\big)\,\mathrm{d}x\mathrm{d}s \\
&- \mathbb{E}\int_0^t \int_{\mathbb{R}^3} (\lambda+\nu)\mathcal{B}[b(\varrho_L)] \cdot \mathrm{div}\mathbf{u}_L\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}s \\
&+ \mathbb{E}\int_0^t \int_{\mathbb{R}^3} a\varrho_L^\gamma\,\nabla\eta \cdot \mathcal{B}[b(\varrho_L)]\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

We now use a sequence of compactly supported smooth functions b_m to approximate $\varrho \mapsto \varrho^\Theta$ to get

$$
\mathbb{E}\int_{\mathbb{R}^3} \eta\big(\varrho_L \mathbf{u}_L\big) \cdot \mathcal{B}\left(\varrho_L^\Theta\right) \,\mathrm{d}x = \mathbb{E}\sum_{i=1}^{13} J_i \qquad (3.64)
$$

where

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{13} J_i := \mathbb{E}&\int_{\mathbb{R}^3} \eta\big(\varrho_{0,L}\mathbf{u}_{0,L}\big)\cdot\mathcal{B}\left[\varrho_{0,L}^{\Theta}\right]\,\mathrm{d}x\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big(\varrho_L\mathbf{u}_L\big)\cdot\,\mathcal{B}\left[\operatorname{div}(\varrho_L^{\Theta}\,\mathbf{u}_L)\right]\mathrm{d}x\,\mathrm{d}s\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big(\varrho_L\mathbf{u}_L\big)\cdot\,\mathcal{B}\left[\Theta\varrho_L^{\Theta}\operatorname{div}\mathbf{u}_L\right]\mathrm{d}x\mathrm{d}s\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big(\varrho_L\mathbf{u}_L\big)\cdot\,\mathcal{B}\left[\varrho_L^{\Theta}\operatorname{div}\mathbf{u}_L\right]\mathrm{d}x\mathrm{d}s\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\big(\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L\big):\nabla\mathcal{B}(\varrho_L^{\Theta})\,\mathrm{d}x\mathrm{d}s\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\nu\nabla\mathbf{u}_L:\nabla\mathcal{B}(\varrho_L^{\Theta})\,\mathrm{d}x\mathrm{d}s\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta(\lambda+\nu)\varrho_L^{\Theta}\,\mathrm{div}\mathbf{u}_L\,\mathrm{d}x\,\mathrm{d}s\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta a\varrho_L^{\gamma+\Theta}\,\mathrm{d}x\,\mathrm{d}s\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \eta\mathcal{B}(\varrho_L^{\Theta})\Phi(\varrho_L,\varrho_L\mathbf{u}_L)\,\mathrm{d}W\,\mathrm{d}x\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} (\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L):\nabla\eta\otimes\mathcal{B}\big(\varrho_L^{\Theta}\big)\,\mathrm{d}x\mathrm{d}s\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} \nu\nabla\mathbf{u}_L:\nabla\eta\otimes\mathcal{B}\big(\varrho_L^{\Theta}\big)\,\mathrm{d}x\mathrm{d}s\\
&-\mathbb{E}\int_0^t\int_{\mathbb{R}^3} (\lambda+\nu)\mathcal{B}(\varrho_L^{\Theta})\cdot\mathrm{div}\mathbf{u}_L\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}s\\
&+\mathbb{E}\int_0^t\int_{\mathbb{R}^3} a\varrho_L^{\gamma}\,\nabla\eta\cdot\mathcal{B}(\varrho_L^{\Theta})\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
\tag{3.65}
$$

To improve the regularity of ϱ, we aim at estimating J_8 in terms of the rest. To do this, we first set the left-hand side of (3.64) to $\mathbb{E}\,J_0$. Then by using (3.48), (3.51), (3.54) and heavy reliance on Hölder inequalities, we can show just as in [16, Propositions 5.1, 6.1] for $\delta = 0$ that

$$
\mathbb{E}\,J_i \lesssim_r 1, \quad \text{for all} \quad i\in\{0,1,\dots,13\}\setminus\{8\}
$$

for some constants $c = c(\Theta,\gamma)$ which are in particular, independent of L.

Indeed having set the left-hand side of (3.64) to $\mathbb{E}\,J_0$, we can use Hölder's inequality, the fact that $\gamma > \frac{3}{2}$ and (3.58) for $q = 2$ to get that

$$\begin{aligned}\mathbb{E}\, J_0 &\leq \mathbb{E}\,\left(\left\|\sqrt{\varrho_L}\right\|_3 \left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2 \left\|\mathcal{B}(\varrho_L^\Theta)\right\|_6\right)\\ &\lesssim \mathbb{E}\,\left(\left\|\varrho_L\right\|_{3/2}^{\frac{1}{2}} \left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2 \left\|\varrho_L^\Theta\right\|_2\right)\\ &\lesssim \mathbb{E}\,\left(\left\|\varrho_L\right\|_{\gamma}^{\frac{1}{2}} \left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2 \left\|\varrho_L^\Theta\right\|_2\right)\end{aligned} \tag{3.66}$$

holds with constants independent of L.

Remark 3.3.6. In (3.66) and the estimates below, we are using the notation

$$\left\|\cdot\right\|_p := \left\|\cdot\right\|_{L^p(\tilde{B}_r)}$$

unless otherwise stated.

And by $(3.48)_1$ and (3.49),

$$\begin{aligned}&\mathbb{E}\,\left(\left\|\varrho_L\right\|_{\gamma}^{\frac{1}{2}} \left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2 \left\|\varrho_L^\Theta\right\|_2\right)\\ &\quad= \mathbb{E}\,\left(\left\|\varrho_L\right\|_{\gamma}^{\frac{1}{2}} \left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2 \left\|\varrho_L\right\|_{2\Theta}^{\Theta}\right)\\ &\quad\leq \left[\left(\mathbb{E}\left\|\varrho_L\right\|_{\gamma}^{\frac{q_1}{2}}\right)^{\frac{1}{q_1}} \left(\mathbb{E}\left\|\sqrt{\varrho_L}\mathbf{u}_L\right\|_2^{q_2}\right)^{\frac{1}{q_2}} \left(\mathbb{E}\left\|\varrho_L\right\|_{2\Theta}^{q_3\Theta}\right)^{\frac{1}{q_3}}\right] \lesssim 1\end{aligned} \tag{3.67}$$

uniformly in L provided

$$2\Theta < \gamma, \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1. \tag{3.68}$$

A similar estimate holds for J_1.

Now for $p = \frac{6\gamma}{5\gamma - 6}$ such that $\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{p} = 1$, we can choose $q = \frac{6\gamma}{7\gamma - 6}$ so that

$p = \frac{6\gamma}{5\gamma-6} = \frac{3q}{3-q}$. It then follows from (3.58) and [29, Theorem 5.2] that

$$\begin{aligned} J_3 + J_4 &= (1-\Theta)\mathbb{E}\int_0^t \int_{\mathbb{R}^3} \eta(\varrho_L \mathbf{u}_L)\cdot \mathcal{B}\left[\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right] \mathrm{d}x\, \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left(\left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\mathcal{B}\left[\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right]\right\|_p\right) \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\nabla\mathcal{B}\left[\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right]\right\|_q \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right\|_q \mathrm{d}s \end{aligned} \tag{3.69}$$

where Hölder's inequality in time and probability further yields

$$\begin{aligned} &\mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right\|_q \mathrm{d}s \\ &\leq \mathbb{E}\left[\left(\sup_{s\in[0,t]}\left\|\varrho_L\right\|_\gamma\right)\left(\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \left\|\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right\|_q^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq \left(\mathbb{E}\sup_{s\in[0,t]}\left\|\varrho_L\right\|_\gamma^{q_1}\right)^{\frac{1}{q_1}} \times \left[\mathbb{E}\left(\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)^{\frac{q_2}{2}}\right]^{\frac{1}{q_2}} \\ &\times \left[\mathbb{E}\left(\int_0^t \left\|\varrho_L^\Theta \operatorname{div} \mathbf{u}_L\right\|_q^2 \mathrm{d}s\right)^{\frac{q_3}{2}}\right]^{\frac{1}{q_3}} \\ &=: \left(J_{3,4}^1 \times J_{3,4}^2 \times J_{3,4}^3\right) \end{aligned} \tag{3.70}$$

where $J_{3,4}^1$ and $J_{3,4}^2$ are uniformly bounded due to (3.49) and $(3.48)_2$. Now note that Hölder's inequality with coefficients satisfying $\frac{1}{2} + \frac{1}{k} = \frac{1}{q}$ holds provided $k = \frac{2q}{2-q}$. Also note that we can choose $q_1 = q_2 = q_3 = 3$ in the previous inequality. In particular, this choice yields $\frac{1}{q_1} + \frac{1}{q_2} = \frac{q_3-1}{q_3} = \frac{2}{3}$ and so

$$\begin{aligned} \left(J_{3,4}^3\right)^3 &\leq \mathbb{E}\left(\int_0^t \left\|\varrho_L^\Theta\right\|_k^2 \left\|\operatorname{div} \mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)^{\frac{3}{2}} \\ &\leq \mathbb{E}\left(\sup_{s\in[0,t]}\left\|\varrho_L^\Theta\right\|_k^2 \int_0^t \left\|\operatorname{div} \mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)^{\frac{3}{2}} \\ &= \mathbb{E}\left(\sup_{s\in[0,t]}\left\|\varrho_L\right\|_{k\Theta}^{2\Theta} \int_0^t \left\|\operatorname{div} \mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)^{\frac{3}{2}} \\ &\leq \left(\mathbb{E}\left[\sup_{s\in[0,t]}\left\|\varrho_L\right\|_{k\Theta}^{2\Theta}\right]^{q_1}\right)^{\frac{3}{2q_1}} \left(\mathbb{E}\left[\int_0^t \left\|\operatorname{div} \mathbf{u}_L\right\|_2^2 \mathrm{d}s\right]^{q_2}\right)^{\frac{3}{2q_2}} \lesssim 1 \end{aligned} \tag{3.71}$$

uniformly in L by (3.49) and (3.51) provided

$$k\Theta = \frac{2q}{2-q}\Theta = \frac{3\gamma}{2\gamma-3}\Theta \leq \gamma. \tag{3.72}$$

Also, we can choose p such that $\frac{1}{\gamma} + \frac{2}{6} + \frac{1}{p} = 1$ in order to gain from (3.58),

$$\begin{aligned} J_5 &\lesssim \mathbb{E}\int_0^t \left[\left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6^2 \left\|\nabla(\mathcal{B}(\varrho_L^\Theta))\right\|_p\right] \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left[\left\|\varrho_L\right\|_\gamma \left\|\nabla\mathbf{u}_L\right\|_2^2 \left\|\varrho_L^\Theta\right\|_p\right] \mathrm{d}s \end{aligned} \tag{3.73}$$

and by Hölder's inequality, (3.49) and (3.51),

$$\begin{aligned} &\mathbb{E}\int_0^t \left[\left\|\varrho_L\right\|_\gamma \left\|\nabla\mathbf{u}_L\right\|_2^2 \left\|\varrho_L^\Theta\right\|_p\right] \mathrm{d}s \\ &\leq \mathbb{E}\left[\left(\sup_{s\in[0,t]}\left\|\varrho_L\right\|_\gamma\right)\left(\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2^2 \mathrm{d}s\right)\left(\sup_{s\in[0,t]}\left\|\varrho_L^\Theta\right\|_p\right)\right] \\ &= \left(\mathbb{E}\sup_{s\in[0,t]}\left\|\varrho_L\right\|_\gamma^{q_1}\right)^{\frac{1}{q_1}} \left(\mathbb{E}\left[\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2^2 \mathrm{d}s\right]^{q_2}\right)^{\frac{1}{q_2}} \left(\mathbb{E}\sup_{s\in[0,t]}\left\|\varrho_L\right\|_{p\Theta}^{q_3\Theta}\right)^{\frac{1}{q_3}} \end{aligned} \tag{3.74}$$

is bounded uniformly in L provided

$$p\Theta = \frac{3\gamma\Theta}{2\gamma-3} < \gamma. \tag{3.75}$$

Similarly, we gain from (3.58)

$$\begin{aligned} J_6 &\lesssim \mathbb{E}\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2 \left\|\nabla(\mathcal{B}(\varrho_L^\Theta))\right\|_2 \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2 \left\|\varrho_L^\Theta\right\|_2 \mathrm{d}s \\ &\lesssim \mathbb{E}\left[\left(\sup_{s\in[0,t]}\left\|\varrho_L\right\|_{2\Theta}^\Theta\right)\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2 \mathrm{d}s\right] \\ &\lesssim \left(\mathbb{E}\sup_{s\in[0,t]}\left\|\varrho_L\right\|_{2\Theta}^{2\Theta}\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^t \left\|\nabla\mathbf{u}_L\right\|_2 \mathrm{d}s\right]^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.76}$$

is uniformly bounded in L if

$$2\Theta < \gamma. \tag{3.77}$$

Young's inequality and (3.77) above gives

$$\begin{aligned} J_7 &\lesssim \mathbb{E}\int_0^t \left(\left\|\operatorname{div}\mathbf{u}_L\right\|_2^2 + \left\|\varrho_L^\Theta\right\|_2^2\right) \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left(\left\|\operatorname{div}\mathbf{u}_L\right\|_2^2 + \left\|\varrho_L\right\|_{2\Theta}^{2\Theta}\right) \mathrm{d}s \\ &\lesssim \mathbb{E}\left(\int_0^t \left\|\operatorname{div}\mathbf{u}_L\right\|_2^2 \mathrm{d}s + \sup_{s\in[0,t]} \left\|\varrho_L\right\|_{2\Theta}^{2\Theta}\right) \lesssim 1. \end{aligned} \tag{3.78}$$

Now observe that $\mathbb{E}(J_9) = 0$ since J_9 is an Itô stochastic integral.

Lastly, to estimate J_2, we use the operator $\mathcal{B}$ in negative spaces which can be found in [52], [9] or [33] in place of (3.58). For the choice of $p = \frac{6\gamma}{5\gamma-6} = \frac{3q}{3-q}$ and $\frac{1}{6}+\frac{1}{r}=\frac{1}{p}$ or $r = \frac{3\gamma}{2\gamma-3}$, we gain from the aforementioned results

$$\begin{aligned} J_2 &\leq \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\mathcal{B}\left[\operatorname{div}(\varrho_L^\Theta \mathbf{u}_L)\right]\right\|_p \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6 \left\|\varrho_L^\Theta \mathbf{u}_L\right\|_p \mathrm{d}s \\ &\lesssim \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left\|\mathbf{u}_L\right\|_6^2 \left\|\varrho_L^\Theta\right\|_r \mathrm{d}s \\ &\lesssim_r \mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left(\left\|\mathbf{u}_L\right\|_2^2 + \left\|\nabla\mathbf{u}_L\right\|_2^2\right) \left\|\varrho_L\right\|_{r\Theta}^\Theta \mathrm{d}s \end{aligned} \tag{3.79}$$

where

$$\begin{aligned} &\mathbb{E}\int_0^t \left\|\varrho_L\right\|_\gamma \left(\left\|\mathbf{u}_L\right\|_2^2 + \left\|\nabla\mathbf{u}_L\right\|_2^2\right) \left\|\varrho_L\right\|_{r\Theta}^\Theta \mathrm{d}s \\ &\leq \mathbb{E}\left(\sup_{s\in[0,t]} \left\|\varrho_L\right\|_\gamma\right) \int_0^t \left(\left\|\mathbf{u}_L\right\|_2^2 + \left\|\nabla\mathbf{u}_L\right\|_2^2\right) \mathrm{d}s \left(\sup_{s\in[0,t]} \left\|\varrho_L\right\|_{r\Theta}^\Theta\right) \\ &\leq \left(\mathbb{E}\sup_{s\in[0,t]} \left\|\varrho_L\right\|_\gamma^{q_1}\right)^{\frac{1}{q_1}} \times \left(\mathbb{E}\left[\int_0^t \left(\left\|\mathbf{u}_L\right\|_2^2 + \left\|\nabla\mathbf{u}_L\right\|_2^2\right) \mathrm{d}s\right]^{q_2}\right)^{\frac{1}{q_2}} \\ &\times \left(\mathbb{E}\sup_{s\in[0,t]} \left\|\varrho_L\right\|_{r\Theta}^{q_3\Theta}\right)^{\frac{1}{q_3}} \end{aligned} \tag{3.80}$$

is uniformly bounded in L provided

$$r\Theta < \gamma \Leftrightarrow \Theta < \frac{2\gamma - 3}{3} \tag{3.81}$$

since $r = \frac{3\gamma}{2\gamma-3}$.

For the same Hölder conjugates used in (3.73), we also obtain by (3.59),

$$\begin{aligned} J_{10} &\lesssim \mathbb{E} \int_0^t \left[\|\varrho_L\|_\gamma \|\mathbf{u}_L\|_6^2 \|\mathcal{B}(\varrho_L^\Theta)\|_\infty \|\nabla\eta\|_p \right] \mathrm{d}s \\ &\lesssim \mathbb{E} \int_0^t \left[\|\varrho_L\|_\gamma \|\nabla\mathbf{u}_L\|_2^2 \|\varrho_L^\Theta\|_p \right] \mathrm{d}s. \end{aligned} \tag{3.82}$$

As such, we obtain the uniform estimate (3.74) under the constraints that $p\Theta \le \gamma$ which is the same as $\Theta \le \frac{2\gamma}{3} - 1$.

To estimate J_{11}, we subsequently follow the approach of J_6 by first using (3.58), i.e.,

$$\begin{aligned} J_{11} &\lesssim \mathbb{E} \int_0^t \|\nabla\mathbf{u}_L\|_2 \|\mathcal{B}(\varrho_L^\Theta)\|_6 \|\nabla\eta\|_3 \,\mathrm{d}s \\ &\lesssim \mathbb{E} \int_0^t \|\nabla\mathbf{u}_L\|_2 \|\varrho_L^\Theta\|_2 \,\mathrm{d}s \\ &\lesssim \mathbb{E} \left[\left(\sup_{s\in[0,t]} \|\varrho_L\|_{2\Theta}^\Theta \right) \int_0^t \|\nabla\mathbf{u}_L\|_2 \,\mathrm{d}s \right] \\ &\lesssim \left(\mathbb{E} \sup_{s\in[0,t]} \|\varrho_L\|_{2\Theta}^{2\Theta} \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_0^t \|\nabla\mathbf{u}_L\|_2 \,\mathrm{d}s \right]^2 \right)^{\frac{1}{2}} \end{aligned} \tag{3.83}$$

is uniformly bounded in L under the condition (3.77).

Now provided that $2\Theta < \gamma$, we also get from (3.58) that,

$$\begin{aligned} J_{12} &\lesssim \mathbb{E} \int_0^t \left[\|\mathcal{B}(\varrho_L^\Theta)\|_6 \|\mathrm{div}\mathbf{u}_L\|_2 \|\nabla\eta\|_3 \right] \mathrm{d}s \\ &\lesssim \mathbb{E} \int_0^t \left[\|\varrho_L^\Theta\|_2 \|\mathrm{div}\mathbf{u}_L\|_2 \right] \mathrm{d}s. \\ &\lesssim \mathbb{E} \int_0^t \left(\|\mathrm{div}\,\mathbf{u}_L\|_2^2 + \|\varrho_L^\Theta\|_2^2 \right) \mathrm{d}s \\ &\lesssim \mathbb{E} \int_0^t \left(\|\mathrm{div}\,\mathbf{u}_L\|_2^2 + \|\varrho_L\|_{2\Theta}^{2\Theta} \right) \mathrm{d}s \\ &\lesssim \mathbb{E} \left(\int_0^t \|\mathrm{div}\,\mathbf{u}_L\|_2^2 \,\mathrm{d}s + \sup_{s\in[0,t]} \|\varrho_L\|_{2\Theta}^{2\Theta} \right) \lesssim 1 \end{aligned} \tag{3.84}$$

uniformly bounded in L under the condition (3.77).

Finally, we consider $p = \frac{3\gamma}{2\gamma-3}$ such that $p\Theta \le \gamma$ and we obtain the following uniform bound

$$
\begin{aligned}
J_{13} &\lesssim \mathbb{E} \int_0^t \left[\left\| \mathcal{B}(\varrho_L^\Theta) \right\|_\infty \left\| \varrho_L^\gamma \right\|_1 \left\| \nabla \eta \right\|_\infty \right] \mathrm{d}s \\
&\lesssim \mathbb{E} \int_0^t \left[\left\| \varrho_L^\Theta \right\|_p \left\| \varrho_L \right\|_\gamma^\gamma \right] \mathrm{d}s. \\
&\lesssim \mathbb{E} \int_0^t \left(\left\| \varrho_L \right\|_{\Theta p}^{2\Theta} + \left\| \varrho_L \right\|_\gamma^{2\gamma} \right) \mathrm{d}s \\
&\lesssim \mathbb{E} \sup_{s \in (0,t)} \left(\left\| \varrho_L \right\|_{\Theta p}^{2\Theta} + \left\| \varrho_L \right\|_\gamma^{2\gamma} \right) \lesssim 1.
\end{aligned}
\tag{3.85}
$$

Taking the intersection of all the conditions (3.68), (3.72), (3.75), (3.77) and (3.81) imposed on all of the variables used in the estimates (3.66), (3.69), (3.71), (3.73), (3.76), (3.78), (3.79), (3.82), (3.83), (3.84) and (3.85) above, we obtain by making J_8 the subject,

$$
\mathbb{E} \left\| \varrho_L \right\|_{L^{\gamma+\Theta}((0,T)\times B^3_{r,L})} \lesssim_r 1 \tag{3.86}
$$

uniformly in L provided $0 < \Theta < \frac{2}{3}\gamma - 1$. □

We now show that not only are our earlier estimates bounded uniformly on the torus $\mathbb{T}_L^3$ but due to the fact that each constants obtained are uniform in L, they are indeed bounded locally on the whole space $\mathbb{R}^3$.

Lemma 3.3.7. For any $L \geq 1$ and for any $1 \leq p < \infty$, we have that

$$
\begin{aligned}
&\mathbb{E} \left| \left(\int_0^T \left\| \mathbf{u}_L \right\|_{W^{1,2}(B_r)}^2 \mathrm{d}t \right)^{\frac{1}{2}} \right|^p \lesssim 1, \\
&\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \sqrt{\varrho_L} \mathbf{u}_L \right\|_{L^2(B_r)} \right|^p \lesssim 1, \\
&\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \varrho_L \right\|_{L^\gamma(B_r)} \right|^p \lesssim_r 1, \\
&\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \varrho_L \mathbf{u}_L \right\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)} \right|^p \lesssim_r 1, \\
&\mathbb{E} \left| \left(\int_0^T \left\| \varrho_L \mathbf{u}_L \otimes \mathbf{u}_L \right\|_{L^{\frac{6\gamma}{4\gamma+3}}(B_r)}^2 \mathrm{d}t \right)^{\frac{1}{2}} \right|^p \lesssim_r 1, \\
&\mathbb{E} \int_0^T \left\| \varrho_L \right\|_{L^{\gamma+\Theta}(B_r)}^{\gamma+\Theta} \mathrm{d}t \lesssim_r 1.
\end{aligned}
\tag{3.87}
$$

uniformly in L for balls $B_r \subset \mathbb{R}^3$ of radius $r > 0$. Furthermore, the initial data also satisfy

$$\begin{aligned}\varrho_{0,L} &\in L^q\big(\Omega; L^\gamma_{\mathrm{loc}}(\mathbb{R}^3)\big),\\ \sqrt{\varrho_{0,L}}\mathbf{u}_{0,L} &\in L^q\big(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^3)\big),\\ \mathbf{m}_{0,L} &\in L^q\big(\Omega; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)\big)\end{aligned} \tag{3.88}$$

uniformly in L for any $2 \leq q < \infty$.

Proof. We will only show the first uniform estimate as the rest can be done in a similar manner in conjunction with (3.48), (3.54), (3.43)–(3.47) and Lemma 3.3.4.

Let $L \in \mathbb{N}$ and let $B_r \subset \mathbb{R}^3$ be the ball of radius $r > 0$ centered at the origin. If $B_r \subset \mathbb{T}^3_L$, then we notice that we can directly deduce from $(3.48)_2$ that

$$\mathbf{u}_L \in L^p\left(\Omega; L^2(0,T; W^{1,2}(B_r))\right) \tag{3.89}$$

uniformly in L. Otherwise, we can use the same argument as in the justification of (3.49) above to get from $(3.48)_2$,

$$\|\mathbf{u}_L\|_{L^p(\Omega;L^2(0,T;W^{1,2}(B_r)))} \leq c(p,r), \quad \forall r > 0 \tag{3.90}$$

uniformly in L.

By combining (3.89) and (3.90), we can deduce that

$$\mathbf{u}_L \in L^p\left(\Omega; L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3))\right) \tag{3.91}$$

uniformly in L. □

Before proceeding further with our compactness arguments, we first recall that by [16, Theorem 2.4], the following renormalized continuity equation

$$\begin{aligned}\int_{\mathbb{R}^3} b(\varrho_L)\,\phi\,\mathrm{d}x &= \int_{\mathbb{R}^3} b\big(\varrho_{0,L}\big)\,\phi(0)\,\mathrm{d}x\\ &+ \int_0^T\int_{\mathbb{R}^3}\big[b(\varrho_L)\mathbf{u}_L\big]\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^T\int_{\mathbb{R}^3}\big[\big(b'(\varrho_L)\varrho_L - b(\varrho_L)\big)\mathrm{div}\,\mathbf{u}_L\big]\,\phi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.92}$$

holds $\mathbb{P}$-a.s. for all $t \in [0,T]$, any $\phi \in C_c^\infty(\mathbb{R}^3)$ and $b \in C^1(\mathbb{R})$ such that $b'(z) = 0$ for all $z \geq M_b$. Note that the family $(\varrho_L, \mathbf{u}_L)$ may be extended by periodicity to the whole space so that (3.92) holds given the existence result in [16, Theorem 2.4].

3.3.8 Compactness

To proceed with compactness, we first introduce the canonical measure ν_L associated to $(\varrho_L, \mathbf{u}_L, \varrho_L\mathbf{u}_L, \nabla\mathbf{u}_L)$. That is, for $\mathfrak{P}(X)$ denoting the set of probability measures on X, we consider the weakly-$*$ measurable mapping

$$\nu_L : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathfrak{P}\big(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}\big) \approx \mathfrak{P}(\mathbb{R}^{16})$$

given by

$$\nu_{L,\omega,t,x}(\cdot) = \delta_{[\varrho_L, \mathbf{u}_L, \varrho_L\mathbf{u}_L, \nabla\mathbf{u}_L](\omega,t,x)}(\cdot).$$

Then per the discussions in [15, Section 2.8], see also [15, Section 4.4.3.1], we may regard ν_L as a random variable taking values in the following space

$$\Big(L^\infty_{w^*}\big((0,T; L^\infty_{w^*,\mathrm{loc}}(\mathbb{R}^3)); \mathfrak{P}(\mathbb{R}^{16})\big), w^*\Big)$$

equipped with the following (functional) weak-$*$ topology

$$L^\infty_{w^*}\big(0,T; L^\infty_{w^*,\mathrm{loc}}(\mathbb{R}^3); \mathfrak{P}(\mathbb{R}^{16})\big) \to \mathbb{R}, \quad \nu \mapsto \int_0^T \int_{\mathbb{R}^3} \psi(t,x) \int_{\mathbb{R}^{16}} \phi(\xi)\,\mathrm{d}\nu_{\omega,t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi \in L^1(0,T, L^1_{\mathrm{loc}}(\mathbb{R}^3))$, for all $\phi \in C_b(\mathbb{R}^{16})$.

We then define the following path space

$$\chi = \chi_{\varrho_0} \times \chi_{\mathbf{m}_0} \times \chi_{\frac{\mathbf{m}_0}{\sqrt{\varrho_0}}} \times \chi_{\mathbf{u}} \times \chi_{\varrho} \times \chi_{\varrho\mathbf{u}} \times \chi_W \times \chi_\nu$$

where

$$
\begin{aligned}
\chi_{\varrho_0} &= L^{\gamma}_{\text{loc}}(\mathbb{R}^3), \quad \chi_{\mathbf{m}_0} = L^{\frac{2\gamma}{\gamma+1}}_{\text{loc}}(\mathbb{R}^3), \quad \chi_{\frac{\mathbf{m}_0}{\sqrt{\varrho_0}}} = L^2_{\text{loc}}(\mathbb{R}^3),\\
\chi_{\mathbf{u}} &= \left(L^2(0,T; W^{1,2}_{\text{loc}}(\mathbb{R}^3)), w\right),\\
\chi_{\varrho} &= C_w\left([0,T]; L^{\gamma}_{\text{loc}}(\mathbb{R}^3)\right) \cap (L^{\gamma+\Theta}(0,T; L^{\gamma+\Theta}_{\text{loc}}(\mathbb{R}^3)), w),\\
\chi_{\varrho\mathbf{u}} &= C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\text{loc}}(\mathbb{R}^3)\right),\\
\chi_W &= C\left([0,T]; \mathfrak{U}_0\right),\\
\chi_{\nu} &= \left(L^{\infty}_{w^*}\left(0,T; L^{\infty}_{w^*,\text{loc}}(\mathbb{R}^3); \mathfrak{P}(\mathbb{R}^{16})\right), w^*\right),
\end{aligned}
$$

and let

1. $\mu_{\varrho_{0,L}}$ be the law of $\varrho_{0,L}$ on χ_{ϱ_0},
2. $\mu_{\mathbf{m}_{0,L}}$ be the law of $\mathbf{m}_{0,L}$ on $\chi_{\mathbf{m}_0}$,
3. $\mu_{\frac{\mathbf{m}_{0,L}}{\sqrt{\varrho_{0,L}}}}$ be the law of $\frac{\mathbf{m}_{0,L}}{\sqrt{\varrho_{0,L}}}$ on $\chi_{\frac{\mathbf{m}_0}{\sqrt{\varrho_0}}}$,
4. $\mu_{\mathbf{u}_L}$ be the law of $\mathbf{u}_L$ on $\chi_{\mathbf{u}}$,
5. μ_{ϱ_L} be the law of ϱ_L on the space χ_{ϱ},
6. $\mu_{\varrho_L\mathbf{u}_L}$ be the law of $\varrho_L\mathbf{u}_L$ on the space $\chi_{\varrho\mathbf{u}}$,
7. μ_W be the law of W on the space χ_W,
8. μ_{ν} be the law of ν_L on the space χ_{ν},
9. μ^L be the joint law of $\mathbf{u}_L$, ϱ_L, $\varrho_L\mathbf{u}_L$, ν_L and W on the space χ.

We now proceed to show tightness of the above family of laws. We start with the laws on the initial data and show the following proposition.

Proposition 3.3.9. The family of measures $\left\{\mu_{\varrho_{0,L}}; L \geq 1\right\}$, $\left\{\mu_{\mathbf{m}_{0,L}}; L \geq 1\right\}$ and $\left\{\mu_{\frac{\mathbf{m}_{0,L}}{\sqrt{\varrho_{0,L}}}}; L \geq 1\right\}$ are tight on χ_{ϱ_0}, $\chi_{\mathbf{m}_0}$ and $\chi_{\frac{\mathbf{m}_0}{\sqrt{\varrho_0}}}$ respectively.

Proof. Since $\chi_{\varrho_0} \times \chi_{\mathbf{m}_0} \times \chi_{\frac{\mathbf{m}_0}{\sqrt{\varrho_0}}}$ is a product of Polish spaces, the proof of this proposition follows from (3.43)–(3.47) and Prokhorov's theorem. □

To show tightness for the family of laws on the velocities, we first show compactness of the following set on the space it is defined.

Proposition 3.3.10. For any $R > 0$, the exist $c(r) > 0$, $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that the set

$$A_R := \Big\{ \mathbf{u} \in L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)) \,:\, \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(B_r))} \leq c(r)R, \quad \forall r \in \mathbb{N} \Big\}.$$

is relatively compact in $\chi_{\mathbf{u}}$

Proof. To see this, fix $R > 0$ and consider the sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset A_R$ so that

$$\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(B_r))} \lesssim_r R, \quad \forall r \in \mathbb{N}$$

holds uniformly in $n \in \mathbb{N}$. Then in particular,

$$\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(B_1))} \lesssim R,$$

holds uniformly for all $n \in \mathbb{N}$. This uniform bound implies that we can find a subsequence $\big(\mathbf{u}_{n_1}\big)_{n\in\mathbb{N}} \subset \big(\mathbf{u}_n\big)_{n\in\mathbb{N}}$ and a limit point $\mathbf{u} \in A_R$ such that

$$\mathbf{u}_{n_1} \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2\big(0,T; W^{1,2}(B_1)\big). \tag{3.93}$$

But since $\mathbf{u} \in A_R$, it means that it is uniformly bounded in $L^2\big(0,T; W^{1,2}(B_2)\big)$ as well. As such, we can find a further subsequence $\big(\mathbf{u}_{n_2}\big)_{n\in\mathbb{N}} \subset \big(\mathbf{u}_{n_1}\big)_{n\in\mathbb{N}}$ and a limit point $\mathbf{v} \in A_R$ such that such that

$$\mathbf{u}_{n_2} \rightharpoonup \mathbf{v} \quad \text{in} \quad L^2\big(0,T; W^{1,2}(B_2)\big). \tag{3.94}$$

This further implies that

$$\mathbf{u}_{n_2} \rightharpoonup \mathbf{v} \quad \text{in} \quad L^2\big(0,T; W^{1,2}(B_1)\big) \tag{3.95}$$

since $B_1 \subset B_2$ and thus by uniqueness of limits, $\mathbf{v} = \mathbf{u}$.

By repeating this argument, we can construct a diagonal family $\{\mathbf{u}_n^n\}_{n\in\mathbb{N}} \subset \{\mathbf{u}_n\}_{n\in\mathbb{N}}$ that is a common subsequence of all possible sequences $\{\mathbf{u}_n^m\}_{n\in\mathbb{N}}$, $m \in \{0\}\cup\mathbb{N}$ where $\mathbf{u}_n^0 := \mathbf{u}_n$, such that up to uniqueness of limits,

$$\mathbf{u}_n^n \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(0,T;W^{1,2}(B_r))$$

for all $r \in \mathbb{N}$. This finishes the proof. $\square$

Proposition 3.3.11. The family of measures $\{\mu_{\mathbf{u}_L};\, L \geq 1\}$ is tight on $\chi_{\mathbf{u}}$.

Proof. To do this, we let $R > 0$, then by Proposition 3.3.10, there exists a compact subset $A_R \subset \chi_{\mathbf{u}}$.

Now since

$$(A_R)^C := \Big\{\mathbf{u} \in L^2(0,T;W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)) : \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(B_r))} > c(r)R, \text{ for some } r > 0\Big\},$$

for any measure $\mu_{\mathbf{u}_L} \in \{\mu_{\mathbf{u}_L};\, L \geq 1\}$, there exists an $r > 0$ such that by Chebyshev's inequality, we obtain

$$\begin{aligned}\mu_{\mathbf{u}_L}\big((A_R)^C\big) &= \mathbb{P}\Big(\|\mathbf{u}_L\|_{L^2(0,T;W^{1,2}(B_r))} > c(r)R\Big)\\ &\leq \frac{1}{c(r)R}\mathbb{E}\left(\|\mathbf{u}_L\|_{L^2(0,T;W^{1,2}(B_r))}\right) \leq \frac{1}{R} \to 0.\end{aligned}$$

as $R \to \infty$, where we have used (3.90) in the last inequality. We have thus shown tightness of $\{\mu_{\mathbf{u}_L};\, L \geq 1\}$ on $\chi_{\mathbf{u}}$. $\square$

Proposition 3.3.12. The family of measures $\{\mu_{\varrho_L};\, L \geq 1\}$ is tight on χ_{ϱ}.

Proof. From the uniform bound $(3.54)_1$, we can deduce that

$$\mathbb{E}\left|\sup_{t\in[0,T]}\big\|\mathrm{div}(\varrho_L\mathbf{u}_L)\big\|_{W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p \lesssim_r 1 \tag{3.96}$$

uniformly in L which implies that

$$\mathbb{E}\left|\sup_{t\in[0,T]}\big\|\partial_t\varrho_L\big\|_{W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p\lesssim_r 1 \tag{3.97}$$

uniformly in L by the use of the continuity equation. Thus, ϱ_L has a Lipschitz continuous representation (not relabelled) such that the estimate

$$\mathbb{E}\,\big\|\varrho_L\big\|^p_{C^{0,1}\left([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\right)}\lesssim_r 1 \tag{3.98}$$

holds uniformly in L.

Furthermore, by combining the uniform bound (3.98) and the estimate (3.49) with the compact embedding (see [87, Corollary B.2])

$$L^\infty\big(0,T;L^\gamma(B_r)\big)\cap C^{0,1}\big([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\big)\hookrightarrow C_w\big([0,T];L^\gamma(B_r)\big), \tag{3.99}$$

we gain tightness in $C_w\big([0,T];L^\gamma(B_r)\big)$. More precisely, if we first consider a similar diagonal argument as in the proof of Proposition 3.3.10, we gain that the set

$$\begin{aligned}C_R:=\Big\{&\varrho\in L^\infty\big(0,T;L^\gamma(B_r)\big)\cap C^{0,1}\big([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\big)\\ &:\ \|\varrho\|_{L^\infty\left(0,T;L^\gamma(B_r)\right)}+\|\varrho\|_{C^{0,1}\left([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\right)}\lesssim_r R\quad\forall r>0\Big\}\end{aligned}$$

is relatively compact in $C_w\big([0,T];L^\gamma(B_r)\big)$. Furthermore since

$$\begin{aligned}(C_R)^C:=\Big\{&\varrho\in L^\infty\big(0,T;L^\gamma(B_r)\big)\cap C^{0,1}\big([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\big)\\ &:\ \|\varrho\|_{L^\infty\left(0,T;L^\gamma(B_r)\right)}+\|\varrho\|_{C^{0,1}\left([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\right)}>c(r)R\ \text{ for some }\ r>0\Big\},\end{aligned}$$

there exists an $r>0$ such that from (3.49) and (3.98), we get that

$$\begin{aligned}&\mathbb{P}\left(\|\varrho_L\|_{L^\infty\left(0,T;L^\gamma(B_r)\right)}>\frac{c(r)R}{2}\right)+\mathbb{P}\left(\|\varrho_L\|_{C^{0,1}\left([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\right)}>\frac{c(r)R}{2}\right)\\ &\lesssim_r\frac{1}{R}\Big(\mathbb{E}\sup_{t\in[0,T]}\big\|\varrho_L\big\|_{L^\gamma(B_r)}+\mathbb{E}\|\varrho_L\|_{C^{0,1}\left([0,T];W^{-1,\frac{2\gamma}{\gamma+1}}(B_r)\right)}\Big)\lesssim\frac{1}{R}\to 0\end{aligned}$$

uniformly in L as $R \to \infty$. The above result and (3.99) finishes the proof of tightness in $C_w\big([0,T]; L^\gamma(B_r)\big)$.

Tightness in $L^{\gamma+\Theta}\big(0,T; L^{\gamma+\Theta}(B_r)\big)$ follows directly from Lemma 3.3.4. $\square$

Lemma 3.3.13. Let $l > \frac{3}{2}$. Then for any $r > 0$ and for any $p \in [1,\infty)$, we have that

$$\overline{Y}^L \in L^p\Big(\Omega; L^2\big(0,T; W^{-l,2}(B_r)\big)\Big),$$

uniformly in L where

$$\overline{Y}^L := \operatorname{div}(\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L) - \nu\Delta\mathbf{u}_L - (\lambda+\nu)\nabla \operatorname{div}\mathbf{u}_L - \nabla\varrho_L^\gamma$$

Proof. Let $r > 0$. Then we can notice from $(3.54)_2$ that,

$$\big\|\operatorname{div}(\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L)\big\|_{L^p\left(\Omega; L^2\left(0,T; W^{-1,\frac{6\gamma}{4\gamma+3}}(B_r)\right)\right)} \lesssim_r 1.$$

However since the embedding

$$L^p\left(\Omega; L^2\left(0,T; W^{-1,\frac{6\gamma}{4\gamma+3}}(B_r)\right)\right) \hookrightarrow L^p\left(\Omega; L^2\left(0,T; W^{-l,2}(B_r)\right)\right)$$

is continuous provided $l > \frac{3(1+\gamma)}{2\gamma} > \frac{3}{2}$, it follows that

$$\operatorname{div}(\varrho_L \mathbf{u}_L \otimes \mathbf{u}_L) \in L^p\left(\Omega; L^2\left(0,T; W^{-l,2}(B_r)\right)\right)$$

uniformly in L for $l > \frac{3}{2}$.

Now from (3.51), we can deduce that

$$\big\|\nu\Delta\mathbf{u}_L + (\lambda+\nu)\nabla\operatorname{div}\mathbf{u}_L\big\|_{L^p\left(\Omega; L^2\left(0,T; W^{-1,2}(B_r)\right)\right)} \lesssim_r 1$$

uniformly in L and from $(3.87)_3$, we again

$$\mathbb{E}\left|\sup_{t\in[0,T]}\left\|\nabla\varrho_L^\gamma\right\|_{W^{-1,1}(B_r)}\right|^p \lesssim_r 1$$

uniformly in L.

We can then use Sobolev's embedding to further deduce that

$$\nabla\varrho_L^\gamma \in L^p\big(\Omega; L^2\left(0,T; W^{-l,2}\left(B_r\right)\right)\big)$$

uniformly in L provided $l > \frac{3}{2}$.

Collecting the estimates above yields the claim. □

Proposition 3.3.14. The set $\left\{\mu_{\varrho_L\mathbf{u}_L};\, L\geq 1\right\}$ is tight on $\chi_{\varrho u}$.

Proof. We decompose $(\varrho_L\mathbf{u}_L)(t)$ into two parts

$$\begin{aligned}(\varrho_L\mathbf{u}_L)(t) &= (\varrho_L\mathbf{u}_L)(0) - \int_0^t \left[\operatorname{div}(\varrho_L\mathbf{u}_L\otimes\mathbf{u}_L) - \nu\Delta\mathbf{u}_L + (\lambda+\nu)\nabla\operatorname{div}\mathbf{u}_L - \nabla\varrho_L^\gamma\right]\mathrm{d}s\\ &\quad + \int_0^t \Phi(\varrho_L, \varrho_L\mathbf{u}_L)\,\mathrm{d}W(s)\\ &=: Y^L(t) + Z^L(t)\end{aligned}$$

where $Z^L(t)$ represents the stochastic forcing part and $Y^L(t)$, the rest.

Now we notice that from Lemma 3.3.13 and Poincaré's inequality in time, we can deduce on a ball B_r of radius $r>0$ that

$$Y^L \in L^p\Big(\Omega; W^{1,2}\big(0,T; W^{-l,2}(B_r)\big)\Big)$$

uniformly in L. And by continuous embedding in time,

$$\left\|Y^L\right\|_{L^p\left(\Omega; C^\vartheta\left([0,T]; W^{-l,2}(B_r)\right)\right)} \lesssim_r 1 \tag{3.100}$$

uniformly in L provided $\vartheta < \frac{1}{2}$.

Now since the diffusion coefficients $\mathbf{g}_k$ are assumed to have compact support in space, see (3.1), we can apply the Burkholder–Davis–Gundy inequality to get for $\theta \geq 1$

$$
\begin{aligned}
\mathbb{E}\big\|Z^L(t) - Z^L(s)\big\|^\theta_{W^{-l,2}(\mathbb{R}^3)} &= \mathbb{E}\left\| \int_s^t \Phi(\varrho_L, \varrho_L \mathbf{u}_L)\,\mathrm{d}W(\sigma)\right\|^\theta_{W^{-l,2}(\mathbb{R}^3)} \\
&\lesssim \mathbb{E}\left(\int_s^t \sum_{k\geq 1} \big\|\Phi(\varrho_L, \varrho_L \mathbf{u}_L)(e_k)\big\|^2_{W^{-l,2}(\mathbb{R}^3)}\,\mathrm{d}\sigma\right)^{\frac{\theta}{2}} \\
&= c\,\mathbb{E}\left(\int_s^t \sum_{k\geq 1} \|\mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L)\|^2_{W^{-l,2}(K)}\,\mathrm{d}\sigma\right)^{\frac{\theta}{2}}
\end{aligned}
\tag{3.101}
$$

for $K \Subset \mathbb{R}^3$, $l > 3/2$ and where the hidden constants only depends on θ.

On the other hand, by an analogous estimate as in (3.6)–(3.7) combined with Hölder's inequality in time, we gain the estimate

$$
\begin{aligned}
\mathbb{E}&\left(\int_s^t \sum_{k\geq 1} \|\mathbf{g}_k(\varrho_L, \varrho_L \mathbf{u}_L)\|^2_{W^{-l,2}(K)}\,\mathrm{d}\sigma\right)^{\frac{\theta}{2}} \\
&\lesssim \mathbb{E}\left(\int_s^t \int_K \big(1 + \varrho_L^\gamma + |\sqrt{\varrho_L}\mathbf{u}_L|^2\big)\,\mathrm{d}x\,\mathrm{d}\sigma\right)^{\frac{\theta}{2}} \\
&\lesssim \mathbb{E}\left(|t-s| \sup_{\sigma\in[s,t]} \int_K \big(1 + \varrho_L^\gamma + |\sqrt{\varrho_L}\mathbf{u}_L|^2\big)\,\mathrm{d}x\right)^{\frac{\theta}{2}} \\
&\lesssim |t-s|^{\frac{\theta}{2}}
\end{aligned}
\tag{3.102}
$$

uniformly in L by using $(3.87)_{2,3}$.

We have therefore shown that the estimate

$$
\mathbb{E}\big\|Z^L(t) - Z^L(s)\big\|^\theta_{W^{-l,2}(\mathbb{R}^3)} \lesssim |t-s|^{\frac{\theta}{2}} \tag{3.103}
$$

holds uniformly in L for $\theta \geq 1$, so that by applying Kolmogorov's continuity criterion, we get

$$
\big\|Z^L\big\|_{L^p\left(\Omega; C^\vartheta\left([0,T]; W^{-l,2}(\mathbb{R}^3)\right)\right)} \lesssim 1 \tag{3.104}
$$

uniformly in L for $\vartheta > 0$ small and where the constant only depends on K and θ.

By combining (3.100) with (3.104), we obtain locally on balls B_r, $r>0$, the estimate

$$\left\|\varrho_L \mathbf{u}_L\right\|_{L^p\left(\Omega; C^\vartheta\left([0,T]; W^{-l,2}(B_r)\right)\right)} \lesssim 1 \tag{3.105}$$

uniformly in L.

Tightness then follows by making use of the above, equation $(3.54)_1$ and the compact embedding

$$L^\infty\big(0,T; L^{\frac{2\gamma}{\gamma+1}}(B_r)\big) \cap C^\vartheta\big([0,T]; W^{-l,2}(B_r)\big) \hookrightarrow C_w\big([0,T]; L^{\frac{2\gamma}{\gamma+1}}(B_r)\big),$$

see [87, Corollary B.2]. □

Proposition 3.3.15. The family of measures $\{\mu_{\nu_L};\, L\geq 1\}$ is tight on χ_ν.

Proof. Firstly, by a similar argument as in Proposition 3.3.10 and [15, Proposition 2.8.5], we can deduce that for any $R>0$, the set

$$\begin{aligned} D_R := \Bigg\{&\nu \in L^\infty_{w^*}\Big(0,T; L^\infty_{w^*,\mathrm{loc}}(\mathbb{R}^3); \mathfrak{P}(\mathbb{R}^{16})\Big) \\ &: \int_0^T\int_{B_r}\int_{\mathbb{R}^{16}} \Big(|\xi_1|^{\gamma+\Theta} + \sum_{i=2}^4 |\xi_i|^{\frac{2\gamma}{\gamma+1}} + \sum_{i=5}^{16}|\xi_i|^2\Big)\,\mathrm{d}\nu_{\omega,t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t \lesssim_r R, \quad \forall r>0\Bigg\}. \end{aligned}$$

is relatively compact in χ_ν given $(3.87)_{1,4,6}$. Thus, there exist an $r>0$ such that

$$\begin{aligned} &\mu_{\nu_L}\Big((D_R)^C\Big) \\ &= \mathbb{P}\Bigg(\int_0^T\int_{B_r}\int_{\mathbb{R}^{16}} \Big(|\xi_1|^{\gamma+\Theta} + \sum_{i=2}^4 |\xi_i|^{\frac{2\gamma}{\gamma+1}} + \sum_{i=5}^{16}|\xi_i|^2\Big)\,\mathrm{d}\nu_{\omega,t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t > c(r)R\Bigg) \\ &= \mathbb{P}\Bigg(\int_0^T\int_{B_r} \Big(|\varrho_L|^{\gamma+\Theta} + |\varrho_L\mathbf{u}_L|^{\frac{2\gamma}{\gamma+1}} + |\mathbf{u}_L|^2 + |\nabla\mathbf{u}_L|^2\Big)\,\mathrm{d}x\,\mathrm{d}t > c(r)R\Bigg) \\ &\lesssim_r \frac{1}{R}\,\mathbb{E}\int_0^T\int_{B_r} \Big(|\varrho_L|^{\gamma+\Theta} + |\varrho_L\mathbf{u}_L|^{\frac{2\gamma}{\gamma+1}} + |\mathbf{u}_L|^2 + |\nabla\mathbf{u}_L|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim_r \frac{1}{R} \to 0 \end{aligned}$$

uniformly in L as $R\to\infty$. □

Proposition 3.3.16. The family of measures $\{\mu^L;\, L\geq 1\}$ is tight on χ.

Proof. This follows from the above tightness results in addition to the fact that μ_W is tight since it is a Radon measure on the Polish space χ_W. $\square$

From Proposition 3.3.16, we cannot immediately use Skorokhod representation theorem to deduce that $\{\mu^L; L \geq 1\}$ is relatively compact (i.e. Prokhorov theorem), since the path space χ is not metrizable. However, we may use instead, the Jakubowski–Skorokhod representation theorem, Theorem 2.4.29 that gives a similar result but for more general spaces including quasi-Polish spaces, see Definition 2.4.23, the space in which most of our functions above live. For example, $\chi_{\varrho\mathbf{u}}$ is not metrizable hence not a Polish space. However, one can verify that it is a quasi-Polish spaces in the sense of Definition 2.4.23.

Moving on, applying Theorem 2.4.29 yields the following result:

Proposition 3.3.17. There exists a subsequence $\mu^n := \mu^{L_n}$ for $n \in \mathbb{N}$, a complete probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ with χ-valued Borel measurable random variables $\Big(\tilde{\varrho}_{0,n}, \tilde{\mathbf{m}}_{0,n}, \frac{\tilde{\mathbf{m}}_{0,n}}{\sqrt{\tilde{\varrho}_{0,n}}}, \tilde{\mathbf{u}}_n, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n, \tilde{W}_n, \tilde{\nu}_n\Big)$, and some corresponding random variables $\Big(\tilde{\varrho}_0, \tilde{\mathbf{m}}_0, \frac{\tilde{\mathbf{m}}_0}{\sqrt{\tilde{\varrho}_0}}, \tilde{\mathbf{u}}, \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{W}, \tilde{\nu}\Big)$ such that

- the law of $\Big(\tilde{\varrho}_{0,n}, \tilde{\mathbf{m}}_{0,n}, \frac{\tilde{\mathbf{m}}_{0,n}}{\sqrt{\tilde{\varrho}_{0,n}}}, \tilde{\mathbf{u}}_n, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n, \tilde{W}_n, \tilde{\nu}_n\Big)$ is given by the law $\mu^n = \mathcal{L}\Big(\varrho_{0,L_n}, \mathbf{m}_{0,L_n}, \frac{\mathbf{m}_{0,L_n}}{\sqrt{\varrho_{0,L_n}}}, \mathbf{u}_{L_n}, \varrho_{L_n}, \varrho_{L_n}\mathbf{u}_{L_n}, W, , \nu_{L_n}\Big)$ for each $n \in \mathbb{N}$,
- the law of $\Big(\tilde{\varrho}_0, \tilde{\mathbf{m}}_0, \frac{\tilde{\mathbf{m}}_0}{\sqrt{\tilde{\varrho}_0}}, \tilde{\mathbf{u}}, \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{W}, \tilde{\nu}\Big)$ on χ, denoted by μ, is a Radon measure,
- $\Big(\tilde{\varrho}_{0,n}, \tilde{\mathbf{m}}_{0,n}, \frac{\tilde{\mathbf{m}}_{0,n}}{\sqrt{\tilde{\varrho}_{0,n}}}, \tilde{\mathbf{u}}_n, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n, \tilde{W}_n, \tilde{\nu}_n\Big)$ converges $\tilde{\mathbb{P}}$-a.s to the random variables

$\left(\tilde{\varrho}_0, \tilde{\mathbf{m}}_0, \frac{\tilde{\mathbf{m}}_0}{\sqrt{\tilde{\varrho}_0}}, \tilde{\mathbf{u}}, \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{W}, \tilde{\nu}\right)$ in the topology of χ as $n \to \infty$, i.e.

$$
\begin{aligned}
&\tilde{\varrho}_{0,n} \to \tilde{\varrho}_0 \text{ in } L^{\gamma}_{\mathrm{loc}}(\mathbb{R}^3) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{\mathbf{m}}_{0,n} \to \tilde{\mathbf{m}}_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\frac{\tilde{\mathbf{m}}_{0,n}}{\sqrt{\tilde{\varrho}_{0,n}}} \to \frac{\tilde{\mathbf{m}}_0}{\sqrt{\tilde{\varrho}_0}} \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^3) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{\varrho}_n \to \tilde{\varrho} \text{ in } C_w\left([0,T]; L^{\gamma}_{\mathrm{loc}}(\mathbb{R}^3)\right) \cap (L^{\gamma+\Theta}(0,T; L^{\gamma+\Theta}_{\mathrm{loc}}(\mathbb{R}^3)), w) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{\mathbf{u}}_n \to \tilde{\mathbf{u}} \text{ in } \left(L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)), w\right) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{\mathbf{m}}_n \to \tilde{\mathbf{m}} \text{ in } C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)\right) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{W}_n \to \tilde{W} \text{ in } C\left([0,T]; \mathfrak{U}_0\right) && \tilde{\mathbb{P}}\text{-a.s.,}\\
&\tilde{\nu}_n \overset{*}{\rightharpoonup} \tilde{\nu} \text{ in } L^{\infty}\big(0,T; L^{\infty}_{\mathrm{loc}}(\mathbb{R}^3); \mathfrak{P}(\mathbb{R}^{16})\big) && \tilde{\mathbb{P}}\text{-a.s.,}
\end{aligned}
\tag{3.106}
$$

Lemma 3.3.18. We have that $\tilde{\mathbf{m}}_n = \tilde{\varrho}_n \tilde{\mathbf{u}}_n$ and $\tilde{\mathbf{m}} = \tilde{\varrho}\tilde{\mathbf{u}}$ hold $\tilde{\mathbb{P}}$-a.s.

Proof. The first part is because the joint laws of

$$
\left(\tilde{\mathbf{u}}_n, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n\right) \quad \text{and} \quad \left(\mathbf{u}_{L_n}, \varrho_{L_n}, \varrho_{L_n}\mathbf{u}_{L_n}\right) \tag{3.107}
$$

coincide.

For the second, because of the convergence $(3.106)_{4,5}$ of $\tilde{\varrho}_n$ and $\tilde{\mathbf{u}}_n$ in χ_ϱ and $\chi_{\mathbf{u}}$ respectively, we have that

$$
\tilde{\varrho}_n \tilde{\mathbf{u}}_n \rightharpoonup \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{in} \quad L^1\big(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^3)\big) \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{3.108}
$$

Subsequently, because of (3.107), the claim follows. □

Furthermore, we can extend (3.106) by showing the following result.

Corollary 3.3.19. The following $\tilde{\mathbb{P}}$-a.s. convergence holds:

$$
\begin{aligned}
\tilde{\varrho}_n \tilde{\mathbf{u}}_n \to \tilde{\varrho}\tilde{\mathbf{u}} \quad &\text{in} \quad L^2\big(0,T; W^{-1,2}(B_r)\big),\\
\tilde{\varrho}_n \tilde{\mathbf{u}}_n \otimes \tilde{\mathbf{u}}_n \rightharpoonup \tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \quad &\text{in} \quad L^1\big(0,T; L^1(B_r)\big),
\end{aligned}
\tag{3.109}
$$

for any $r > 0$.

Proof. To show the convergence in $(3.109)_1$, we first observe that by the use $(3.106)_6$ and the continuous embedding

$$C_w\big([0,T]; L^{\frac{2\gamma}{\gamma+1}}(B_r)\big) \hookrightarrow L^2\big(0,T; W^{-1,2}(B_r)\big)$$

which holds for any $r > 0$, it follows that

$$\tilde{\varrho}_n \tilde{\mathbf{u}}_n \to \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{in} \quad L^2\big(0,T; W^{-1,2}(B_r)\big) \tag{3.110}$$

$\tilde{\mathbb{P}}$-a.s. finishing the proof of $(3.109)_1$.

Now with (3.110) in hands, we gain $(3.109)_2$ by a weak-strong pairing with $(3.106)_5$. This is so because given (3.110) and $(3.106)_5$, we obtain for any bounded matrix $\mathbb{A} \in L^\infty\big((0,T) \times B_r\big)$,

$$\int_0^T \int_{B_r} \tilde{\varrho}_n \tilde{\mathbf{u}}_n \otimes \tilde{\mathbf{u}}_n : \mathbb{A} \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{B_r} \tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \mathbb{A}\,\mathrm{d}x\,\mathrm{d}t \tag{3.111}$$

$\tilde{\mathbb{P}}$-a.s. which is equivalent to $(3.109)_2$. □

To extend this new probability space $\big(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}\big)$ into a stochastic basis and to ensure that the corresponding stochastic integral is well-defined, we now aim to construct a family of non-anticipative filtrations. Recall Definition 2.4.12 and refer to [15, Remark 2.3.7] for why this is required.

To do the above construction, we first collect the following family of $\tilde{\mathbb{P}}$-augmented canonical filtrations

$$\begin{aligned}
\sigma_t[\tilde{\varrho}_n] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\varrho}_n(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T], \\
\sigma_t[\tilde{\mathbf{u}}_n] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\mathbf{u}}_n(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T], \\
\sigma_t[\tilde{\beta}^n_k] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\beta}^n_k(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T]
\end{aligned} \tag{3.112}$$

for each $n \in \mathbb{N}$. And for the limit random variable, the following corresponding set of natural filtrations

$$
\begin{aligned}
\sigma_t[\tilde{\varrho}] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\varrho}(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T], \\
\sigma_t[\tilde{\mathbf{u}}] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\mathbf{u}}(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T], \\
\sigma_t[\tilde{\beta}_k] &= \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\beta}_k(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T].
\end{aligned} \tag{3.113}
$$

Given these canonical filtrations, we can now construct and endow the probability space with the following pair of filtrations

$$
\begin{aligned}
\tilde{\mathscr{F}}_t^n &= \sigma\left(\sigma_t[\tilde{\varrho}_n] \cup \sigma_t[\tilde{\mathbf{u}}_n] \cup \bigcup_{k=1}^{\infty} \sigma_k\left[\tilde{\beta}_k^n\right]\right), \quad t \in [0,T] \\
\tilde{\mathscr{F}}_t &= \sigma\left(\sigma_t[\tilde{\varrho}] \cup \sigma_t[\tilde{\mathbf{u}}] \cup \bigcup_{k=1}^{\infty} \sigma_k\left[\tilde{\beta}_k\right]\right), \quad t \in [0,T]
\end{aligned} \tag{3.114}
$$

on the family of sequences $(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \tilde{W}_n)$ and the limit random variables $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W})$ respectively. The fact that (3.114) are non-anticipative with respect to their corresponding Wiener processes will be justified below.

With the above preparation, we gain the following lemma.

Lemma 3.3.20. For any $n \in \mathbb{N}$, $[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t^n)_{t\ge 0}, \tilde{\mathbb{P}}), \tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \tilde{W}_n]$ is a weak martingale solution of (1.16) in the sense of Definition 3.2.5 with initial law Λ.

Proof. We first note that $\left(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t^n)_{t\ge 0}, \tilde{\mathbb{P}}\right)$ is a stochastic basis with a complete right-continuous filtration. This follows from Proposition 3.3.17 and the construction of $(3.114)_1$ above.

Now we claim that the process $\tilde{W}_n$ is an $(\tilde{\mathscr{F}}_t^n)$-cylindrical Wiener process and that this filtration $\tilde{\mathscr{F}}_t^n$ as constructed in $(3.114)_1$, is non-anticipative with respect to $\tilde{W}_n$. This is so because from Proposition 3.3.17, since the law of $(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \tilde{\mathbf{m}}_n, \tilde{W}_n)$ coincide with μ^n, it follows from Theorem 2.4.31 that for each $n \in \mathbb{N}$, the process

$$
\tilde{W}_n = \sum_{k\in\mathbb{N}} e_k \tilde{\beta}_k^n
$$

is a cylindrical Wiener process with respect to the corresponding filtration

$$\tilde{\mathscr{F}}_t^n = \sigma\bigg(\sigma_t[\tilde{\varrho}_n] \cup \sigma_t[\tilde{\mathbf{u}}_n] \cup \bigcup_{k=1}^{\infty} \sigma_k\big[\tilde{\beta}_k^n\big] \bigg), \quad t \in [0,T]. \tag{3.115}$$

By definition of a cylindrical Wiener process, it implies that for any such $n \in \mathbb{N}$, the filtration $\tilde{\mathscr{F}}_t^n$ is independent of $\sigma\big(\tilde{W}_n(t+s)-\tilde{W}_n(t)\big)$ for any $s > 0$, which is precisely the notion of $\tilde{\mathscr{F}}_t^n$ being non-anticipative with respect to $\tilde{W}_n$, recall Definition 2.4.12.

Now by using $(3.106)_{1,4}$ and (3.108), we gain that for all $\psi \in C_c^\infty(\mathbb{R}^3)$ and all $t \in [0,T]$, the equation

$$\big\langle \tilde{\varrho}_n(t), \psi \big\rangle - \big\langle \tilde{\varrho}_{0,n}, \psi \big\rangle - \int_0^t \big\langle \tilde{\varrho}_n \tilde{\mathbf{u}}_n, \nabla\psi \big\rangle \, \mathrm{d}s = 0 \tag{3.116}$$

holds $\tilde{\mathbb{P}}$-a.s. This is a consequence of Theorem 2.4.31.

Similarly for all $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$ and all $t \in [0,T]$, we can use $(3.106)_{2,4,5}$, Lemma 3.3.18 and $(3.109)_2$ to gain

$$\begin{aligned}
\big\langle \tilde{\varrho}_n \tilde{\mathbf{u}}_n(t) \,,\, \boldsymbol{\phi} \big\rangle &= \big\langle \tilde{\mathbf{m}}_{0,n} \,,\, \boldsymbol{\phi} \big\rangle + \int_0^t \big\langle \tilde{\mathbf{m}}_n \otimes \tilde{\mathbf{u}}_n \,,\, \nabla\boldsymbol{\phi} \big\rangle \, \mathrm{d}s \\
&- \nu \int_0^t \big\langle \nabla\tilde{\mathbf{u}}_n \,,\, \nabla\boldsymbol{\phi} \big\rangle \, \mathrm{d}s - (\lambda+\nu) \int_0^t \big\langle \operatorname{div}\tilde{\mathbf{u}}_n \,,\, \operatorname{div}\boldsymbol{\phi} \big\rangle \, \mathrm{d}s \\
&+ \int_0^t \big\langle \tilde{\varrho}_n^\gamma \,,\, \operatorname{div}\boldsymbol{\phi} \big\rangle \, \mathrm{d}s + \int_0^t \big\langle \Phi(\tilde{\varrho}_n, \tilde{\mathbf{m}}_n) \mathrm{d}\tilde{W}_n \,,\, \boldsymbol{\phi} \big\rangle
\end{aligned} \tag{3.117}$$

$\tilde{\mathbb{P}}$-a.s. □

Proposition 3.3.21. The local-in-space limit velocity processes in $(3.106)_5$ satisfies $\tilde{\mathbf{u}} \in L^2(0,T;D^{1,2}(\mathbb{R}^3))$ $\tilde{\mathbb{P}}$-a.s.

Proof. Let $B_r \subset \mathbb{R}^3$ be an arbitrary ball of radius $r > 0$. Then from $(3.106)_5$, we have that for $\tilde{\mathbb{P}}$- a.s.,

$$\tilde{\mathbf{u}}_n \rightharpoonup \tilde{\mathbf{u}} \quad \text{in} \quad L^2(0,T;W^{1,2}(B_r)), \text{ for } r > 0.$$

However, lower semicontinuity of norms means that for any such $r > 0$,

$$\|\chi_{B_r}\nabla\tilde{\mathbf{u}}\|_{L^2(0,T;L^2(\mathbb{R}^3))} = \|\nabla\tilde{\mathbf{u}}\|_{L^2(0,T;L^2(B_r))} \leq \liminf_{n\to\infty}\|\nabla\tilde{\mathbf{u}}_n\|_{L^2(0,T;L^2(B_r))}$$

$\tilde{\mathbb{P}}$-a.s. with a right-hand side that is uniformly bounded in $r > 0$, c.f. $(3.87)_1$. Passing to the limit $r \to \infty$ on either side of this inequality with the help of Fatou's lemma shows that $\nabla\tilde{\mathbf{u}} \in L^2(0,T;L^2(\mathbb{R}^3))$. □

Lemma 3.3.22. The process $\tilde{W}$ is an $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process and there exists

$$\overline{p} := \overline{p(\tilde{\varrho})} \in L^{\frac{\gamma+\Theta}{\gamma}}\big((0,T)\times K\big), \quad K \Subset \mathbb{R}^3 \tag{3.118}$$

which are weak limits of $p(\tilde{\varrho}_n)$ satisfying the limit system

$$\begin{aligned} \mathrm{d}\tilde{\varrho} + \mathrm{div}(\tilde{\varrho}\tilde{\mathbf{u}})\mathrm{d}t = 0 \\ \mathrm{d}(\tilde{\varrho}\tilde{\mathbf{u}}) + \big[\mathrm{div}(\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}) - \nu\Delta\tilde{\mathbf{u}} - (\lambda+\nu)\nabla\mathrm{div}\tilde{\mathbf{u}} + \nabla\overline{p}\big]\mathrm{d}t = \overline{\Phi(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})}\,\mathrm{d}\tilde{W} \end{aligned} \tag{3.119}$$

$\tilde{\mathbb{P}}$-a.s. in the sense of distributions. Furthermore, $\overline{\Phi(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})}$ is an $W^{-l,2}(\mathbb{R}^3)$-valued $(\tilde{\mathscr{F}}_t)$-progressively measurable process such that

$$\overline{\Phi(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})} \in L^2\big(0,T;L_2(\mathfrak{U}_0;W^{-l,2}(K))\big), \quad K \Subset \mathbb{R}^3. \tag{3.120}$$

Proof. We start by showing that the limit Wiener process $\tilde{W}$ from Proposition 3.3.17 is non-anticipative with respect to the filtration $(3.114)_2$.

Having shown Lemma 3.3.20, we have that $\tilde{W}_n$ is an $(\mathscr{F}_t^n)$-cylindrical Wiener process and that $(\mathscr{F}_t^n)$ is non-anticipative with respect to $\tilde{W}_n$.

By Lemma 2.4.32 and $(3.106)_7$, we are able to pass to the limit $n \to \infty$ and gain that

$$\tilde{\mathscr{F}}_t = \sigma\left(\sigma_t[\tilde{\varrho}]\,,\,\sigma_t[\tilde{\mathbf{u}}]\,,\,\bigcup_{k=1}^{\infty}\sigma_k\big[\tilde{\beta}_k\big]\right), \quad t\in[0,T] \tag{3.121}$$

is non-anticipative with respect to $\tilde{W}$.

Now since the union of the canonical filtrations $(3.113)_3$, which are contained in (3.121), are the minimal filtration on which the process $\tilde{W}$ is adapted, it follows from Lemma 2.4.33 and Corollary 2.4.34 that $\tilde{W}$ is an $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process.

Now we notice that the identification of the distributional form (3.119) is immediate (except for the pressure and stochastic terms) given Proposition 3.3.17 and in particular, (3.106).

The existence of (3.118) is a direct consequence of $(3.106)_{4b}$ so that in fact, we are able to easily identify the pressure term in the distributional form (3.119) as well.

Now, if we consider the function $\mathbf{g}_k(x, \varrho, \mathbf{m}) \; : \; K \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ for $k \in \mathbb{N}$ and where $K \Subset \mathbb{R}^3$ is the support of the noise term according to (3.1), then

- $\mathbf{g}_k(\cdot, \varrho, \mathbf{m}) : K \to \mathbb{R}$ is measurable for all $(\varrho, \mathbf{m}) \in \mathbb{R}_+ \times \mathbb{R}^3$;
- $\mathbf{g}_k(x, \cdot, \cdot) : \mathbb{R}_+ \to \mathbb{R}$ is continuous for a.e. $x \in K$;

and thus for each $k \in \mathbb{N}$, the function $\mathbf{g}_k := \mathbf{g}_k(x, \varrho, \mathbf{m})$ is a Carathéodory function. Furthermore, since $\varrho \lesssim 1 + \varrho^\gamma$ and $\frac{2\gamma}{\gamma+1} > 1$, we obtain from (3.2),

$$|\mathbf{g}_k(x, \varrho, \mathbf{m})| \lesssim 1 + \varrho^\gamma + |\mathbf{m}|^{\frac{2\gamma}{\gamma+1}}$$

uniformly in $x \in K$, thus by Lemma 2.5.1, there exists a Young measure ν_x generating a function $\overline{\mathbf{g}_k(x, \tilde{\varrho}, \tilde{\mathbf{m}})} = \langle \tilde{\nu}_x, \mathbf{g}_k \rangle$ such that as $n \to \infty$,

$$\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) \rightharpoonup \overline{\mathbf{g}_k(x, \tilde{\varrho}, \tilde{\mathbf{m}})} \quad \text{in} \quad L^r\big((0,T) \times K\big) \tag{3.122}$$

$\tilde{\mathbb{P}}$-a.s. provided $1 < r \leq \frac{\gamma+\Theta}{\gamma} \wedge \frac{\gamma+1}{\gamma}$ where the term $\gamma + \Theta$ follows from $(3.106)_{4b}$.

We now adapt the arguments in the proof of [15, Proposition 4.4.12.] to gain strong convergence of the noise term. To do this, we first observe that we have

$$\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) - \mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}) \to 0 \quad \text{in} \quad L^2\big(0, T; W^{-l,2}(K)\big) \tag{3.123}$$

$\tilde{\mathbb{P}}$-a.s. for $l > \frac{3}{2}$ since for all $\boldsymbol{\varphi} \in C^\infty(K)$, we obtain from (3.2) (recall that $\tilde{\mathbf{m}}_n = \tilde{\varrho}_n \tilde{\mathbf{u}}_n$) that

$$\begin{aligned}
&\int_K \eta\big(\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) - \mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}})\big) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\
&\qquad \lesssim_k \|\boldsymbol{\varphi}\|_{L^\infty(K)} \int_K \eta \tilde{\varrho}_n |\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}| \,\mathrm{d}x \\
&\qquad \lesssim_k \|\tilde{\varrho}_n\|^{\frac{1}{2}}_{L^1(K)} \big\|\eta \tilde{\varrho}_n |\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}|^2\big\|^{\frac{1}{2}}_{L^1(K)}
\end{aligned} \tag{3.124}$$

where $\eta \in C_0^\infty(\mathbb{R}^3)$, $\eta \equiv 1$ in K and

$$\big\|\eta \tilde{\varrho}_n |\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}|^2\big\|_{L^1(K)} = \int_K \eta \tilde{\varrho}_n |\tilde{\mathbf{u}}_n|^2 \,\mathrm{d}x - 2\int_K \eta \tilde{\varrho}_n \tilde{\mathbf{u}}_n \cdot \tilde{\mathbf{u}} \,\mathrm{d}x - \int_K \eta \tilde{\varrho}_n |\tilde{\mathbf{u}}|^2 \,\mathrm{d}x$$

converges strongly to zero in $L^2(0,T)$ due to $(3.87)_2$ and Proposition 3.3.17.

Now since

$$\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) = \big(\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) - \mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}})\big) + \mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}), \tag{3.125}$$

It follows from (3.123) that

$$\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n) \to \overline{\mathbf{g}_k(x, \tilde{\varrho}, \tilde{\mathbf{m}})} \quad \text{in} \quad L^2\big(0,T; W^{-l,2}(K)\big) \tag{3.126}$$

$\tilde{\mathbb{P}}$-a.s. if

$$\mathbf{g}_k(x, \tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}) \to \overline{\mathbf{g}_k(x, \tilde{\varrho}, \tilde{\mathbf{m}})} \quad \text{in} \quad L^2\big(0,T; W^{-l,2}(K)\big) \tag{3.127}$$

$\tilde{\mathbb{P}}$-a.s.

To show (3.127), we first observe that due to (3.92) and Proposition 3.3.17 we gain from Theorem 2.4.31 that

$$\begin{aligned}
\int_{\mathbb{R}^3} b(\tilde{\varrho}_n(t))\, \phi \,\mathrm{d}x &= \int_{\mathbb{R}^3} b(\tilde{\varrho}_{0,n})\, \phi \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} b(\tilde{\varrho}_n) \tilde{\mathbf{u}}_n \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{R}^3} \big(b(\tilde{\varrho}_n) - b'(\tilde{\varrho}_n)\tilde{\varrho}_n\big) \,\mathrm{div}\tilde{\mathbf{u}}_n\, \phi \,\mathrm{d}x \,\mathrm{d}s,
\end{aligned} \tag{3.128}$$

holds $\tilde{\mathbb{P}}$-a.s. for $b \in C_b^1(\mathbb{R})$ such that $b'(z) = 0$ for all $z \geq M_b$.

Given that b is bounded, we obtain from Lemma 2.5.1 in combination with Proposition 3.3.17 that there exists a function $\overline{b(\tilde{\varrho})} = \langle \nu_x, b \rangle$ where ν_x is a Young measure such that as $n \to \infty$,

$$b(\tilde{\varrho}_n) \rightharpoonup \overline{b(\tilde{\varrho})} \quad \text{in} \quad L^p\big((0,T) \times K\big), \tag{3.129}$$

$\tilde{\mathbb{P}}$-a.s. for all $1 < p \leq \infty$. So in particular,

$$b(\tilde{\varrho}_n) \in L^\infty(0,T,L^2(K)) \tag{3.130}$$

uniformly and

$$b(\tilde{\varrho}_n) \rightharpoonup \overline{b(\tilde{\varrho})} \quad \text{in} \quad L^\infty(0,T,L^2(K)), \tag{3.131}$$

$\tilde{\mathbb{P}}$-a.s. Furthermore, we can deduce from (3.128) that

$$\partial_t b(\tilde{\varrho}_n) \in L^2\big(0,T;W^{-1,2}(K)\big) \tag{3.132}$$

with a constant that may depend on $\tilde{\Omega}$. The boundedness (3.132) holds since b is bounded and $\tilde{\mathbf{u}}_n \in L^2(0,T;W^{1,2}(K))$ $\tilde{\mathbb{P}}$-a.s.. By Aubin–Lions lemma, it follows from (3.130) and (3.132) that

$$b(\tilde{\varrho}_n) \to \overline{b(\tilde{\varrho})} \quad \text{in} \quad L^2\big(0,T;W^{-1,2}(K)\big), \tag{3.133}$$

$\tilde{\mathbb{P}}$-a.s. Note the ordinarily, (3.133) may have to hold for a subsequence depending on $\tilde{\Omega}$ when Aubin–Lions lemma is applied. However, since (3.131) holds for the original sequence, (3.133) necessarily holds for the original sequence as well.

The weak and strong convergence, (3.131) and (3.133) respectively thus gives

$$b(\tilde{\varrho}_n) \to \overline{b(\tilde{\varrho})} \quad \text{in} \quad C_w\big([0,T];L^2(K)\big). \tag{3.134}$$

Now given that $\tilde{\mathbf{u}} \in L^2(0,T;W^{1,2}(K))$ $\tilde{\mathbb{P}}$-a.s. and that the embedding

$$C_w\big([0,T];L^2(K)\big) \hookrightarrow L^2\big(0,T;W^{-1,2}(K)\big)$$

is continuous, it follows from (3.134) that

$$b(\tilde{\varrho}_n)B(\tilde{\mathbf{u}}) \to \overline{b(\tilde{\varrho})B(\tilde{\mathbf{u}})} \quad \text{in} \quad L^2\big(0,T;W^{-1,2}(K)\big) \tag{3.135}$$

$\tilde{\mathbb{P}}$-a.s. for any Lipschitz functions b and B. Thus we obtain (3.127) by approximating $\mathbf{g}_k(x,\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})$ by finite sums $\sum_i b_i(\tilde{\varrho})B_i(\tilde{\mathbf{u}})$. We have thus shown that

$$\mathbf{g}_k(x,\tilde{\varrho}_n,\tilde{\mathbf{m}}_n) \to \overline{\mathbf{g}_k(x,\tilde{\varrho},\tilde{\mathbf{m}})} \quad \text{in} \quad L^2\big(0,T;W^{-l,2}(K)\big) \tag{3.136}$$

$\tilde{\mathbb{P}}$-a.s. provided $l > \frac{3}{2}$. So if we set

$$\overline{\Phi(\tilde{\varrho},\tilde{\mathbf{m}})}e_k := \overline{\mathbf{g}_k(x,\tilde{\varrho},\tilde{\mathbf{m}})}, \tag{3.137}$$

then we gain from the summability of the constants in (3.2)–(3.5),

$$\Phi(\tilde{\varrho}_n,\tilde{\mathbf{m}}_n) \to \overline{\Phi(\tilde{\varrho},\tilde{\mathbf{m}})} \quad \text{in} \quad L^2\big(0,T;L_2\big(\mathfrak{U}_0;W^{-l,2}(K)\big)\big) \tag{3.138}$$

$\tilde{\mathbb{P}}$-a.s. for some $l \in \mathbb{R}$.

Given (3.138) and $(3.106)_7$, we are able to apply Lemma 2.4.35 in other to identify the stochastic integral in (3.119). This finishes our proof. $\square$

3.4 Identification of the pressure limit

This section is devoted to showing strong convergence of the density sequence which in turns, enables us to pass to the limit in the nonlinear pressure sequence to gain a corresponding isentropic pressure limit term. Recall from the previous section that uniform boundedness of the quantity $p(\tilde{\varrho}_n) = \tilde{\varrho}_n^\gamma$ in some norm only establishes the existence of a quantity $\overline{p}$ to which the aforementioned sequence convergences weakly to. We therefore aim to show that $\overline{p}$ is in fact isentropic, i.e., $\overline{p} = \tilde{\varrho}^\gamma$, with $\tilde{\varrho}$ being the strong limit of $\tilde{\varrho}_n$ in a suitable space.

To do the above requires some preparation, the first of which is to study the *effective viscous flux* below.

3.4.1 Derivation of the effective viscous flux identity

The main aim of this section is to study compactness for the quantity

$$\tilde{\varrho}_n^\gamma - (\lambda + 2\nu)\mathrm{div}\, \tilde{\mathbf{u}}_n \tag{3.139}$$

which for somewhat ironic reasons, enjoys better regularity properties than the pressure $\tilde{\varrho}_n^\gamma$ alone. Subsequently, (3.139) will help us identify the limit pressure.

To study (3.139) however, we first introduce some operators and collect some preliminary results.

Lemma 3.4.2. The process $(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n)$ is a renormalized solution of the continuity equation with $b = T_k$ defined by

$$T_k(t) = \begin{cases} t & \text{if } 0 \leq t < k, \\ k & \text{if } k \leq t < \infty. \end{cases} \tag{3.140}$$

That is for T_k as defined above, we have that for all $t \in [0, T]$ and for all $\phi \in C_c^\infty(\mathbb{R}^3)$, the equation

$$\begin{aligned} \int_{\mathbb{R}^3} T_k(\tilde{\varrho}_n(t))\, \phi \,\mathrm{d}x = \int_{\mathbb{R}^3} T_k(\tilde{\varrho}_{0,n})\, \phi \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} T_k(\tilde{\varrho}_n)\tilde{\mathbf{u}}_n \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}s \\ - \int_0^t \int_{\mathbb{R}^3} (T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n)\, \mathrm{div}\tilde{\mathbf{u}}_n\, \phi \,\mathrm{d}x \,\mathrm{d}s, \end{aligned} \tag{3.141}$$

holds $\tilde{\mathbb{P}}$-a.s.

Proof. Due to (3.92) and Proposition 3.3.17, the result follows from Theorem 2.4.31. □

Lemma 3.4.3. Let $B_r \subset \mathbb{R}^3$ be a ball or radius $r > 0$. For any $1 < p < \infty$ and $k > 0$ fixed, the following

$$T_k(\tilde{\varrho}_n) \to \overline{T_k(\tilde{\varrho})} \quad \text{in} \quad C_w\big([0,T]; L^p(B_r)\big), \tag{3.142}$$

holds $\tilde{\mathbb{P}}$-a.s. (see (2.28) for the definition of the T_ks.)

Proof. By definition, T_k is a Carathéodory function for each $k \in \mathbb{N}$ and bounded by the constant $k > 0$, i.e. $|T_k(\tilde{\varrho}_n)| < k$. This information can be reformulated as $|T_k(\tilde{\varrho}_n)| \lesssim_k 1 + \tilde{\varrho}_n^0$ so by Lemma 2.5.1, there exists a function $\overline{T_k(\tilde{\varrho})} = \langle \nu_x, T_k \rangle$ where ν_x is a Young measure such that as $n \to \infty$,

$$T_k(\tilde{\varrho}_n) \rightharpoonup \overline{T_k(\tilde{\varrho})} \quad \text{in} \quad L^p\big((0,T) \times B_r\big), \tag{3.143}$$

$\tilde{\mathbb{P}}$-a.s. for all $1 < p \leq \infty$. So in particular,

$$T_k(\tilde{\varrho}_n) \in L^\infty\big(0, T, L^p(B_r)\big), \tag{3.144}$$

$\tilde{\mathbb{P}}$-a.s. for all $1 < p < \infty$.

Furthermore, by Lemma 3.4.2, it follows from a similar argument as in (3.132) that

$$\partial_t T_k(\tilde{\varrho}_n) \in L^2\big(0, T; W^{-1,2}(K)\big) \tag{3.145}$$

and thus the claim follows. $\square$

Lemma 3.4.4. Let T_k be as defined in (3.140) and also let B_r be a ball of radius $r > 0$. Then we have that

$$\big[T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n - T_k(\tilde{\varrho}_n)\big] \operatorname{div} \tilde{\mathbf{u}}_n \to \overline{[T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho})] \operatorname{div} \tilde{\mathbf{u}}} \tag{3.146}$$

$\tilde{\mathbb{P}}$-a.s. in $\big(L^q((0,T) \times B_r), w\big)$ for some $q > 1$.

Proof. First of all, we note that similar to the proof of (3.122), the function $H\big(\tilde{\varrho}, \tilde{u}\big) :=$

$\big(T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}$ is a Carathéodory function satisfying the identity

$$\begin{aligned}
\big|\big(T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}\big| &= \big|\big(T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}\,\mathbb{1}_{\{\tilde\varrho<k\}} \\
&\quad + \big(T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}\,\mathbb{1}_{\{\tilde\varrho\geq k\}}\big| \\
&= \big|\big((T_k)_+'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}\,\mathbb{1}_{\{\tilde\varrho\geq k\}}\big| \\
&= \big|k\operatorname{div}\tilde{\mathbf{u}}\,\mathbb{1}_{\{\tilde\varrho\geq k\}}\big|.
\end{aligned}$$

Furthermore, by Young's inequality, we gain for some $0<\frac{\gamma}{\gamma-1}<3$ that the growth condition

$$\big|\big(T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)\big)\operatorname{div}\tilde{\mathbf{u}}\big| \leq \big|\tilde\varrho\operatorname{div}\tilde{\mathbf{u}}\big| \lesssim \big|\tilde\varrho\big|^{\gamma} + \big|\operatorname{div}\tilde{\mathbf{u}}\big|^{\frac{\gamma}{\gamma-1}} \tag{3.147}$$

holds uniformly in (t,x) and thus, there exists a Young measure $\tilde\nu_{t,x}$ and a limit function satisfying $\overline{[T_k'(\tilde\varrho)\tilde\varrho - T_k(\tilde\varrho)]\operatorname{div}\tilde{\mathbf{u}}} = \langle\tilde\nu_{t,x}, H\rangle$ such that for all

$$1<q\leq\frac{\gamma+\Theta}{\gamma}\wedge\frac{2(\gamma-1)}{\gamma}$$

the convergence (3.146) holds as $n\to\infty$. □

Lemma 3.4.5. Let $\phi:=\phi(x)\in C_c^\infty(\mathbb{R}^3)$. Then for all $1<q_1<\infty$ and $1\leq q_2<\infty$, the following

$$T_k(\tilde\varrho_n)\to\overline{T_k(\tilde\varrho)} \qquad \text{in } L^2\big([0,T];W^{-1,2}(B_r)\big), \tag{3.148}$$

$$\mathcal{A}_i[\phi T_k(\tilde\varrho_n)]\to\mathcal{A}_i[\phi\overline{T_k(\tilde\varrho)}] \qquad \text{in } L^{q_2}\big(0,T;L^{q_1}(\mathbb{R}^3)\big) \tag{3.149}$$

holds $\tilde{\mathbb{P}}$-a.s. for at least some subsequences where $\mathcal{A}_i:=\Delta_{\mathbb{R}^3}^{-1}\partial_i$, see Section 2.2.

Remark 3.4.6. Notice that since the approximate quantities in (3.109) and elsewhere are only defined locally in space, to apply this globally defined operators $\mathcal{A}_i$, it is essentially to pre-multiply our functions by some smooth function $\phi\in C_c^\infty(\mathbb{R})$.

Proof of Lemma 3.4.5. For (3.148), we first note that the embedding

$$C_w\big([0,T];L^p(B_r)\big) \hookrightarrow L^2\big(0,T;W^{-1,2}(B_r)\big)$$

is continuous for $p > \frac{6}{5}$. This information and (3.142) yields the claim (3.148).

Now using (3.142) and $(2.27)_1$, we have that for $t \in [0,T]$, the convergence

$$\mathcal{A}_i[\phi T_k(\tilde{\varrho}_n(t))] \rightharpoonup \mathcal{A}_i[\phi \overline{T_k(\tilde{\varrho}(t))}] \quad \text{in} \quad W^{1,p}(B_r) \quad \forall 1 < p < \infty \tag{3.150}$$

holds $\tilde{\mathbb{P}}$-a.s.. Moreover since the embedding $W^{1,p} \hookrightarrow L^{q_1}$ is compact for any $1 < q_1 < 3p/3 - p$, we conclude $\tilde{\mathbb{P}}$-a.s. that for $t \in [0,T]$,

$$\mathcal{A}_i[\phi T_k(\tilde{\varrho}_n(t))] \to \mathcal{A}_i[\phi \overline{T_k(\tilde{\varrho}(t))}] \quad \text{in} \quad L^{q_1}(B_r) \quad \forall 1 < q_1 < \infty. \tag{3.151}$$

Recall that by definition, T_k is bounded by the constant $k > 0$ and as such, form any $p \in (1,\infty)$ and any $q \in (1,\infty)$, the inequality

$$\sup_n \left[\tilde{\mathbb{E}}\left(\sup_{t\in[0,T]} \big\|T_k(\tilde{\varrho}_n)\big\|^{pq}_{L^p(B_r)}\right)\right] \lesssim k \tag{3.152}$$

holds uniformly in n. Given the uniform integrability (3.152), it follows from Vitali's convergence that for any $1 < q_1 < \infty$,

$$\mathcal{A}_i[\phi T_k(\tilde{\varrho}_n)] \to \mathcal{A}_i[\phi \overline{T_k(\tilde{\varrho})}] \quad \text{in} \quad L^{q_2}(0,T;L^{q_1}(B_r) \quad \forall 1 \leq q_2 < \infty \tag{3.153}$$

$\tilde{\mathbb{P}}$-a.s.. This shows (3.149) by extension by zero outside the support of ϕ. □

Lemma 3.4.7. Let $1/p + 1/q = 1/r < 1$ with $1 < p,q < \infty$ and suppose that

$$v_n \rightharpoonup v, \quad w_n \rightharpoonup w$$

in $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ respectively. Then

$$v_n\mathcal{R}_{ij}[w_n] - w_n\mathcal{R}_{ij}[v_n] \rightharpoonup v\mathcal{R}_{ij}[w] - w\mathcal{R}_{ij}[v]$$

in $L^r(\mathbb{R}^3)$, $i,j=1,2,3$ where $\mathcal{R}_{ij}=\partial_i\mathcal{A}_j=\partial_i\Delta^{-1}_{\mathbb{R}^3}\partial_j$, see Section 2.2.

For the proof of Lemma 3.4.7, see [86, Lemma 4.25] and [43, Lemma 3.4]. The following lemma, Lemma 3.4.8, follows similar arguments in [86, Section 7].

Lemma 3.4.8. Let $\phi^1(x),\phi^2(x)\in C_c^\infty(\mathbb{R}^3)$. Then the following

$$\begin{aligned}
&\mathcal{A}_i[\phi^2(T_k'(\tilde{\varrho}_n)\,\tilde{\varrho}_n-T_k(\tilde{\varrho}_n))\mathrm{div}\,\tilde{\mathbf{u}}_n]\\
&\rightharpoonup \mathcal{A}_i[\phi^2\overline{(T_k'(\tilde{\varrho})\,\tilde{\varrho}-T_k(\tilde{\varrho}))\,\mathrm{div}\,\tilde{\mathbf{u}}}] && \text{in } L^2\big(0,T;W^{1,2}(\mathbb{R}^3)\big)\\
&\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}_n\tilde{u}_n^j]\,\phi^2T_k(\tilde{\varrho}_n)-\phi^1\,\tilde{\varrho}_n\tilde{u}_n^j\,\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n)]\\
&\to \mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}\tilde{u}^j]\,\phi^2\overline{T_k(\tilde{\varrho})}-\phi^1\,\tilde{\varrho}\tilde{u}^j\,\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho})}] && \text{in } L^2\big(0,T;W^{-1,2}(\mathbb{R}^3)\big)\\
&\mathcal{A}_i[\phi^1\,\tilde{\varrho}_n\tilde{u}_n^i]\,T_k(\tilde{\varrho}_n)\partial_j\phi^2-\tilde{\varrho}_n\tilde{u}_n^i\,\mathcal{A}_i[\phi^2T_k(\tilde{\varrho}_n)]\partial_j\phi^1\\
&\to \mathcal{A}_i[\phi^1\,\tilde{\varrho}\tilde{u}^i]\,\overline{T_k(\tilde{\varrho})}\partial_j\phi^2-\tilde{\varrho}\tilde{u}^i\,\mathcal{A}_i[\phi^2\overline{T_k(\tilde{\varrho})}]\partial_j\phi^1 && \text{in } L^2\big((0,T)\times\mathbb{R}^3\big)
\end{aligned}\tag{3.154}$$

holds $\tilde{\mathbb{P}}$-a.s.

Proof. The convergence $(3.154)_1$ follows directly from Lemma 3.4.4 and $(2.27)_1$.

Let us now have a look at $(3.154)_2$. We recall that since $\mathcal{R}_{ij}=\partial_i\mathcal{A}_j$, we can conclude from the compact embedding $L^p(\mathbb{R}^3)\hookrightarrow W^{-1,2}(\mathbb{R}^3)$, $p>\frac{6}{5}$ and (3.150) that for a.a. $t\in[0,T]$

$$\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n(t))]\to\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho}(t))}]\quad\text{in}\quad W^{-1,2}(\mathbb{R}^3)\tag{3.155}$$

$\tilde{\mathbb{P}}$-a.s. And similar to (3.153), we gain from (3.155),

$$\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n)]\to\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho})}]\quad\text{in}\quad L^{p_1}\big(0,T;W^{-1,2}(\mathbb{R}^3)\big)\tag{3.156}$$

$\tilde{\mathbb{P}}$-a.s. for any $1\le p_1<\infty$ by applying Vitali's convergence theorem.

On the other hand, given the regularities in (3.142) and $(3.106)_6$ (also recall Lemma 3.3.18), we gain from Lemma 3.4.7 that for any $s>1$ such that $1>\frac{1}{s}=\frac{1}{p}+\frac{\gamma+1}{2\gamma}$, the following convergence

$$\begin{aligned}&\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n(t)]\,\phi^2T_k(\tilde{\varrho}_n(t))-\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n(t)\,\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n(t))]\\&\qquad\rightharpoonup\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}\tilde{u}^j(t)]\,\phi^2\overline{T_k(\tilde{\varrho}(t))}-\phi^1\,\tilde{\varrho}\tilde{u}^j(t)\,\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho}(t))}]\qquad\text{in}\quad L^s(\mathbb{R}^3)\end{aligned}$$

holds $\tilde{\mathbb{P}}$-a.s. for $t\in[0,T]$. The compact embedding $L^s(\mathbb{R}^3)\hookrightarrow W^{-1,2}(\mathbb{R}^3)$ therefore yields for $t\in[0,T]$,

$$\begin{aligned}&\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n(t)]\,\phi^2T_k(\tilde{\varrho}_n(t))-\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n(t)\,\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n(t))]\\&\qquad\to\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}\tilde{u}^j(t)]\,\phi^2\overline{T_k(\tilde{\varrho}(t))}-\phi^1\,\tilde{\varrho}\tilde{u}^j(t)\,\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho}(t))}]\qquad\text{in}\quad W^{-1,2}(\mathbb{R}^3)\end{aligned}$$

$\tilde{\mathbb{P}}$-a.s.

Now for $\mathrm{d}\tilde{\mathbb{P}}\otimes\mathrm{d}t$ a.e. $(\omega,t)\in\tilde{\Omega}\times[0,T]$, the quantities $\big(\phi\tilde{\varrho}_n\tilde{u}^j_n\big)$ and $\big(\phi T_k(\tilde{\varrho}_n)\big)$ are uniformly bounded in $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ (restricted for the purpose of embedding to $6<p<\infty$) respectively by $(3.106)_6$ and (3.142) (together with Lemma 3.3.18). Thus, we may infer that for $\mathrm{d}\tilde{\mathbb{P}}\otimes\mathrm{d}t$ a.e. $(\omega,t)\in\tilde{\Omega}\times[0,T]$, the quantity

$$\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n]\,\phi^2T_k(\tilde{\varrho}_n)-\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n\,\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n)]$$

is uniformly bounded in $L^s(\mathbb{R}^3)$, $(\gamma+1)/2\gamma+1/p=1/s<1$ by virtue of $(2.27)_1$. With this information, the compact embedding $L^s(\mathbb{R}^3)\hookrightarrow W^{-1,2}(\mathbb{R}^3)$, $s>6/5$ combined with Vitali's convergence theorem yields (up to subsequences),

$$\begin{aligned}&\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n]\,\phi^2T_k(\tilde{\varrho}_n)-\phi^1\,\tilde{\varrho}_n\tilde{u}^j_n\,\mathcal{R}_{ij}[\phi^2T_k(\tilde{\varrho}_n)]\\&\qquad\to\mathcal{R}_{ij}[\phi^1\,\tilde{\varrho}\tilde{u}^j]\,\phi^2\overline{T_k(\tilde{\varrho})}-\phi^1\,\tilde{\varrho}\tilde{u}^j\,\mathcal{R}_{ij}[\phi^2\overline{T_k(\tilde{\varrho})}]\quad\text{in }L^{p_2}\big(0,T;W^{-1,2}(\mathbb{R}^3)\big)\end{aligned}\tag{3.157}$$

$\tilde{\mathbb{P}}$-a.s. for any $1\leq p_2<\infty$. So $(3.154)_2$ follow.

Lastly, as in the proof of $(3.154)_2$, we gain $(3.154)_3$. Recall that $\mathcal{R}_{ji}=\partial_j\mathcal{A}_i$. □

We now aim to finally identify the limit of (3.139). To do this, we perform a similar computation as in (3.64). That is, we apply Itô's formula to the function $f(g_n,\tilde{m}^i_n)=\int_{\mathbb{R}^3}\tilde{m}^i_n\cdot\phi^1(x)\mathcal{A}_i[\phi^2(x)g_n]\,\mathrm{d}x$ where $g_n=T_k(\tilde{\varrho}_n)$ solves the renormalized continuity equation and $\tilde{m}_n^i = \tilde{\varrho}_n \tilde{u}_n^i$, $i = 1, 2, 3$.

Remark 3.4.9. To make the product in the function f above rigorous, it is important to have a preliminary regularization step in the spirit of (3.60)–(3.62). Subsequently, by passing to the limit in the regularization parameter, we gain the following:

$$
\begin{aligned}
\tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i \left[\phi^2 T_k(\tilde{\varrho}_n) \right] \, \mathrm{d}x &= \tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i(0) \, \mathcal{A}_i \left[\phi^2 T_k(\tilde{\varrho}_n(0)) \right] \, \mathrm{d}x \\
& - \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i [\phi^2 \, \partial_j (T_k(\tilde{\varrho}_n) \tilde{u}_n^j)] \mathrm{d}x \, \mathrm{d}s \\
& - \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i \left[\phi^2 \, (T_k'(\tilde{\varrho}_n) \, \tilde{\varrho}_n - T_k(\tilde{\varrho}_n)) \operatorname{div} \tilde{\mathbf{u}}_n \right] \mathrm{d}x \, \mathrm{d}s \\
& + \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \tilde{\varrho}_n \tilde{u}_n^i \tilde{u}_n^j \, \partial_j (\phi^1 \, \mathcal{A}_i [\phi^2 T_k(\tilde{\varrho}_n)]) \, \mathrm{d}x \, \mathrm{d}s \\
& - \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \partial_j \left[\phi^1 \, \mathcal{A}_i \big(\phi^2 T_k(\tilde{\varrho}_n) \big) \right] \partial_j \tilde{u}_n^i \, \mathrm{d}x \, \mathrm{d}s \\
& + \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} [a \tilde{\varrho}_n^\gamma - (\lambda + \nu) \operatorname{div} \tilde{\mathbf{u}}_n] \, \partial_i \big(\phi^1 \mathcal{A}_i [\phi^2 \, T_k(\tilde{\varrho}_n)] \big) \, \mathrm{d}x \, \mathrm{d}s \\
& =: \tilde{\mathbb{E}} \sum_{k=1}^{6} J_k, \quad i = 1, 2, 3.
\end{aligned} \tag{3.158}
$$

where T_k replaces b in the definition of the renormalized equation defined in (3.20).

Remark 3.4.10. Notice that since the noise term is a martingale, it vanishes when we take its expectation, as martingales are constant on average.

Now notice that by integration by parts and the use of the properties of the operators

$\mathcal{A}_i$ and $\mathcal{R}_{ij} = \partial_i \mathcal{A}_j$ (see Section 2.2 for further details), we have that

$$
\begin{aligned}
J_2 &= -\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \Big(\mathcal{R}_{ij} \left[\phi^1 \, \tilde{\varrho}_n \tilde{u}_n^j \right] \, \tilde{u}_n^i \, \phi^2 \, T_k(\tilde{\varrho}_n) + \mathcal{A}_i \left[\phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \right] \, \tilde{u}_n^j \, T_k(\tilde{\varrho}_n) \partial_j \phi^2 \Big) \mathrm{d}x \, \mathrm{d}s \\
J_4 &= \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \Big(\tilde{\varrho}_n \tilde{u}_n^i \tilde{u}_n^j \, \mathcal{A}_i [\phi^2 T_k(\tilde{\varrho}_n)] \partial_j \phi^1 + \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \tilde{u}_n^j \, \mathcal{R}_{ij} [\phi^2 T_k(\tilde{\varrho}_n)] \Big) \, \mathrm{d}x \, \mathrm{d}s \\
J_5 &= \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} (\partial_j \partial_j \phi^1) \, \tilde{u}_n^i \, \mathcal{A}_i \, [\phi^2 T_k(\tilde{\varrho}_n)] \, \mathrm{d}x \, \mathrm{d}s + \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^1 \, \partial_i \, [\phi^2 T_k(\tilde{\varrho}_n)] \, \tilde{u}_n^i \, \mathrm{d}x \, \mathrm{d}s \\
&= \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} (\partial_j \partial_j \phi^1) \, \tilde{u}_n^i \, \mathcal{A}_i \, [\phi^2 T_k(\tilde{\varrho}_n)] \, \mathrm{d}x \, \mathrm{d}s \\
&\quad - \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \Big(\phi^1 \phi^2 \, T_k(\tilde{\varrho}_n) \, \mathrm{div} \, \tilde{\mathbf{u}}_n + \phi^2 \, \tilde{u}_n^i \, T_k(\tilde{\varrho}_n) \, \partial_i \phi^1 \Big) \, \mathrm{d}x \, \mathrm{d}s \\
J_6 &= \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} [a \tilde{\varrho}_n^\gamma - (\lambda + \nu) \mathrm{div} \, \tilde{\mathbf{u}}_n] \, \Big(\mathcal{A}_i [\phi^2 \, T_k(\tilde{\varrho}_n)] \, \partial_i \phi^1 + \phi^1 \phi^2 \, T_k(\tilde{\varrho}_n) \Big) \, \mathrm{d}x \, \mathrm{d}s
\end{aligned}
$$

so that (3.158) becomes

$$
\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \left[a \tilde{\varrho}_n^\gamma - (\lambda + 2\nu) \mathrm{div} \, \tilde{\mathbf{u}}_n \right] \phi^1 \phi^2 \, T_k(\tilde{\varrho}_n) \, \mathrm{d}x \, \mathrm{d}s = \tilde{\mathbb{E}} \sum_{k=1}^{8} I_k \tag{3.159}
$$

where for $i = 1, 2, 3,$

$$
\begin{aligned}
\tilde{\mathbb{E}} \sum_{k=1}^{8} I_k &:= \tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i \left[\phi^2 \, T_k(\tilde{\varrho}_n) \right] \, \mathrm{d}x \\
&- \tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i(0) \, \mathcal{A}_i \left[\phi^2 \, T_k(\tilde{\varrho}_n(0)) \right] \, \mathrm{d}x \\
&+ \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^2 \, \tilde{u}_n^i \, T_k(\tilde{\varrho}_n) \, \partial_i \phi^1 \, \mathrm{d}x \, \mathrm{d}s \\
&- \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} [a \tilde{\varrho}_n^\gamma - (\lambda + \nu) \mathrm{div} \, \tilde{\mathbf{u}}_n] \mathcal{A}_i [\phi^2 \, T_k(\tilde{\varrho}_n)] \, \partial_i \phi^1 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i \left[\phi^2 \, \left(T_k'(\tilde{\varrho}_n) \, \tilde{\varrho}_n - T_k(\tilde{\varrho}_n) \right) \mathrm{div} \, \tilde{\mathbf{u}}_n \right] \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \tilde{u}_n^i \, \left(\mathcal{R}_{ij} [\phi^1 \, \tilde{\varrho}_n \tilde{u}_n^j] \, \phi^2 \, T_k(\tilde{\varrho}_n) - \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^j \, \mathcal{R}_{ij} [\phi^2 \, T_k(\tilde{\varrho}_n)] \right) \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \tilde{u}_n^j \, \left(\mathcal{A}_i [\phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i] \, T_k(\tilde{\varrho}_n) \partial_j \phi^2 \, - \tilde{\varrho}_n \tilde{u}_n^i \, \mathcal{A}_i [\phi^2 T_k(\tilde{\varrho}_n)] \partial_j \phi^1 \right) \, \mathrm{d}x \, \mathrm{d}s \\
&- \nu \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} (\partial_j \partial_j \phi^1) \, \tilde{u}_n^i \, \mathcal{A}_i \, [\phi^2 T_k(\tilde{\varrho}_n)] \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned} \tag{3.160}
$$

Remark 3.4.11. If we set the left-hand side of (3.159) to $\tilde{\mathbb{E}} I_0$, then we point the reader to the difference in the viscosity constant in I_0 and I_4.

Similarly for the limit processes, we obtain from (3.119),

$$\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} [\overline{p} - (\lambda + 2\nu)\mathrm{div}\tilde{\mathbf{u}}]\, \phi^1 \phi^2\, \overline{T_k(\tilde{\varrho})}\, \mathrm{d}x\, \mathrm{d}s = \tilde{\mathbb{E}} \sum_{k=1}^{8} K_k \tag{3.161}$$

where for $i = 1, 2, 3$,

$$\begin{aligned}
\tilde{\mathbb{E}} \sum_{k=1}^{8} K_k &= \tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1\, \tilde{\varrho}\tilde{u}^i\, \mathcal{A}_i \left[\phi^2\, \overline{T_k(\tilde{\varrho})}\right] \mathrm{d}x \\
&- \tilde{\mathbb{E}} \int_{\mathbb{R}^3} \phi^1\, \tilde{\varrho}\tilde{u}^i(0)\, \mathcal{A}_i \left[\phi^2\, \overline{T_k(\tilde{\varrho}(0))}\right] \mathrm{d}x \\
&+ \nu\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^2\, \tilde{u}^i\, \overline{T_k(\tilde{\varrho})}\, \partial_i \phi^1\, \mathrm{d}x\, \mathrm{d}s \\
&- \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} [\overline{p} - (\lambda + \nu)\mathrm{div}\tilde{\mathbf{u}}]\mathcal{A}_i[\phi^2\, \overline{T_k(\tilde{\varrho})}]\, \partial_i \phi^1\, \mathrm{d}x\, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \phi^1\, \tilde{\varrho}\tilde{u}^i\, \mathcal{A}_i \left[\phi^2\, \overline{(T_k'(\tilde{\varrho})\, \tilde{\varrho} - T_k(\tilde{\varrho}))\, \mathrm{div}\, \tilde{\mathbf{u}}}\right] \mathrm{d}x\, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \tilde{u}^i \left(\mathcal{R}_{ij}[\phi^1\, \tilde{\varrho}\tilde{u}^j]\, \phi^2\, \overline{T_k(\tilde{\varrho})} - \phi^1\, \tilde{\varrho}\tilde{u}^j\, \mathcal{R}_{ij}[\phi^2\, \overline{T_k(\tilde{\varrho})}]\right) \mathrm{d}x\, \mathrm{d}s \\
&+ \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \tilde{u}^j \left(\mathcal{A}_i[\phi^1\, \tilde{\varrho}\tilde{u}^i]\, \overline{T_k(\tilde{\varrho})}\partial_j \phi^2 \; - \tilde{\varrho}\tilde{u}^i\, \mathcal{A}_i[\phi^2 \overline{T_k(\tilde{\varrho})}]\partial_j \phi^1\right) \mathrm{d}x\, \mathrm{d}s \\
&- \nu\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} (\partial_j \partial_j \phi^1)\, \tilde{u}^i\, \mathcal{A}_i\, [\phi^2 \overline{T_k(\tilde{\varrho})}]\, \mathrm{d}x\, \mathrm{d}s
\end{aligned} \tag{3.162}$$

and where a 'bar' above a function represents the limit of the corresponding approximate sequence of that functions.

Finally, we state and prove the main result in this section. In the following lemma, we identify (3.161) as the limit process derived from the family of equations solving (3.159). That is, we show that

Lemma 3.4.12. For any $\phi^1(x), \phi^2(x) \in C_c^\infty(\mathbb{R}^3)$, we have that

$$\begin{aligned}
\lim_{n\to\infty} \tilde{\mathbb{E}} \int_0^T \int_{\mathbb{R}^3} [a\tilde{\varrho}_n^\gamma - (\lambda + 2\nu)\mathrm{div}\, \tilde{\mathbf{u}}_n]\, \phi^1 \phi^2 T_k(\tilde{\varrho}_n)\, \mathrm{d}x\, \mathrm{d}t \\
= \tilde{\mathbb{E}} \int_0^T \int_{\mathbb{R}^3} [\overline{p} - (\lambda + 2\nu)\mathrm{div}\, \tilde{\mathbf{u}}]\, \phi^1 \phi^2 \overline{T_k(\tilde{\varrho})}\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned} \tag{3.163}$$

Proof. Combining the strong convergence (3.151) with the weak convergence $(3.106)_6$

and Lemma 3.3.18, it follows by duality pairing that for $t \in [0,T]$,

$$\int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i(t) \mathcal{A}_i[\phi^2 T_k(\tilde{\varrho}_n(t))] \, \mathrm{d}x \to \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho} \tilde{u}^i(t) \mathcal{A}_i[\phi^2 \overline{T_k(\tilde{\varrho}(t))}] \, \mathrm{d}x, \tag{3.164}$$

$\tilde{\mathbb{P}}$-a.s. for $i = 1,2,3$. Furthermore, given the boundedness of T_k and the continuity property of $\mathcal{A}_i$, it follows from Hölder's inequality, $(3.87)_4$ and the equality of laws established in Proposition 3.3.17 that

$$\begin{aligned} &\sup_{n \in \mathbb{N}} \tilde{\mathbb{E}} \left| \int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i(t) \cdot \mathcal{A}_i[\phi^2 T_k(\tilde{\varrho}_n(t))] \, \mathrm{d}x \right|^p \\ &\lesssim_k \sup_{n \in \mathbb{N}} \tilde{\mathbb{E}} \left| \sup_{t \in [0,T]} \left\| \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i \right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)} \right|^p \lesssim 1 \end{aligned} \tag{3.165}$$

for any finite $p > 1$. The inequality (3.165) implies higher uniform integrability (in the sense of probability) of

$$\int_{\mathbb{R}^3} \phi^1 \, \tilde{\varrho}_n \tilde{u}_n^i(t) \mathcal{A}_i[\phi^2 T_k(\tilde{\varrho}_n(t))] \, \mathrm{d}x$$

which together with $\tilde{\mathbb{P}}$-a.s. convergence ensures that convergence $\tilde{\mathbb{E}}\,(I_k) \to \tilde{\mathbb{E}}\,(K_k)$, $k = 1,2$. Thus, we have shown that

$$\tilde{\mathbb{E}}\,(I_1 + I_2) \to \tilde{\mathbb{E}}\,(K_1 + K_2).$$

Similarly, we can use the following weak-strong duality pairings:

$$\begin{aligned} &\{(3.106)_5, (3.148)\}, \\ &\{(3.106)_{5,4b}, (3.149)\}, \\ &\{(3.154)_1, (3.109)_1\}, \\ &\{(3.106)_5, (3.154)_2\}, \\ &\{(3.106)_5, (3.154)_3\}, \\ &\{(3.106)_5, (3.149)\}, \end{aligned}$$

to get that $\tilde{\mathbb{E}}\, I_3 \to \tilde{\mathbb{E}}\, K_3$, $\tilde{\mathbb{E}}\, I_4 \to \tilde{\mathbb{E}}\, K_4$, $\tilde{\mathbb{E}}\, I_5 \to \tilde{\mathbb{E}}\, K_5$, $\tilde{\mathbb{E}}\, I_6 \to \tilde{\mathbb{E}}\, K_6$, $\tilde{\mathbb{E}}\, I_7 \to \tilde{\mathbb{E}}\, K_7$ and

$\tilde{\mathbb{E}}\, I_8 \to \tilde{\mathbb{E}}\, K_8$ respectively. □

3.4.13 Boundedness of the oscillation defect measure

Let $Q = (0,T) \times \mathbb{R}^3$. Showing that indeed $\overline{p} = a\tilde{\varrho}^\gamma$ or equivalently that $\tilde{\varrho}_n \to \tilde{\varrho}$ strongly in $L^p(\tilde{\Omega} \times Q)$ for all $p \in [1, \gamma + \Theta)$ follows Feireisl's approach via the use of the so-called *oscillation defect measure* introduced in [34]. This measures the amplitude of oscillations in the sequence $\tilde{\varrho}_n$ and is a purely deterministic argument even in our stochastic setting. We give this in the lemma below.

Lemma 3.4.14. Let $\tilde{\varrho}$ be the weak limit of the sequence $\tilde{\varrho}_n$ in a suitable topology. Then for any $\phi(x) \in C_c^\infty(\mathbb{R}^3)$ and any $\gamma > \frac{3}{2}$, the following estimate holds

$$\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi\, \left|T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})\right|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t \lesssim 1 \tag{3.166}$$

where the constant does not depend on k.

Remark 3.4.15. In analogy to the deterministic case, the quantity

$$\begin{aligned} &\mathbf{osc}_{\gamma+1}\left[\tilde{\varrho}_n \to \tilde{\varrho}\right](\tilde{\Omega} \times Q) \\ &= \sup_{k\geq 1}\left(\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi\, \left|T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})\right|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t\right) \end{aligned} \tag{3.167}$$

refers to the *oscillation defect measure.*

Proof of 3.4.14. First of all, we note that the identity

$$\begin{aligned} &\lim_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \left(a\tilde{\varrho}_n^\gamma\, T_k(\tilde{\varrho}_n) - \overline{p}\,\overline{T_k(\tilde{\varrho})}\right) \mathrm{d}x\,\mathrm{d}t \\ &\qquad = (\lambda + 2\nu) \lim_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \left[\operatorname{div}\tilde{\mathbf{u}}_n\, T_k(\tilde{\varrho}_n) - \operatorname{div}\tilde{\mathbf{u}}\,\overline{T_k(\tilde{\varrho})}\right] \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.168}$$

holds by virtue of Lemma 3.4.12.

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, \Big(a\tilde{\varrho}_n^\gamma \, T_k(\tilde{\varrho}_n) - \overline{p}\,\overline{T_k(\tilde{\varrho})} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
= \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, a\,(\tilde{\varrho}_n^\gamma - \tilde{\varrho}^\gamma)\,(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})) \, \mathrm{d}x \, \mathrm{d}t \\
+ \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, (\overline{p} - a\tilde{\varrho}^\gamma) \Big(T_k(\tilde{\varrho}) - \overline{T_k(\tilde{\varrho})} \Big) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{3.169}$$

Now since the map $t \mapsto t^\gamma$ is convex while $t \mapsto T_k(t)$ is concave on the interval $(0,\infty)$, the last term in (3.169) above is negative whereas by (2.32),

$$\begin{aligned}
\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, a\,(\tilde{\varrho}_n^\gamma - \tilde{\varrho}^\gamma)\,(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})) \, \mathrm{d}x \, \mathrm{d}t \\
\geq \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, \big| T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \big|^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{3.170}$$

By combining (3.168)–(3.170), we gain

$$\begin{aligned}
\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi \, \big| T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \big|^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}t \\
\leq (\lambda + 2\nu) \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi \, \Big[\operatorname{div} \tilde{\mathbf{u}}_n \, T_k(\tilde{\varrho}_n) - \operatorname{div} \tilde{\mathbf{u}} \, \overline{T_k(\tilde{\varrho})} \Big] \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{3.171}$$

where $\phi = \phi^1\phi^2$.

However, the use of the Hölder and triangle inequalities further yields

$$\begin{aligned}
&\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, \Big[\operatorname{div} \tilde{\mathbf{u}}_n \, T_k(\tilde{\varrho}_n) - \operatorname{div} \tilde{\mathbf{u}} \, \overline{T_k(\tilde{\varrho})} \Big] \, \mathrm{d}x \, \mathrm{d}t \\
&= \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \, \Big[T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) + T_k(\tilde{\varrho}) - \overline{T_k(\tilde{\varrho})} \Big] \operatorname{div} \tilde{\mathbf{u}}_n \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \limsup_{n\to\infty} \tilde{\mathbb{E}} \, \big\| \phi^2 \operatorname{div} \tilde{\mathbf{u}}_n \big\|_{L^2(Q)} \tilde{\mathbb{E}} \Big(\big\| \phi^1 \big(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \big) \big\|_{L^2(Q)} \\
&+ \big\| \phi^1 \big(T_k(\tilde{\varrho}) - \overline{T_k(\tilde{\varrho})} \big) \big\|_{L^2(Q)} \Big).
\end{aligned} \tag{3.172}$$

And given (3.142), we can use the lower semi-continuity of norms to obtain the inequality

$$\begin{aligned}
\tilde{\mathbb{E}} \, \Big\| \phi^1 \big(T_k(\tilde{\varrho}) - \overline{T_k(\tilde{\varrho})} \big) \Big\|_{L^2(Q)} &\leq \liminf_{n\to\infty} \tilde{\mathbb{E}} \, \big\| \phi^1 \big(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \big) \big\|_{L^2(Q)} \\
&\leq \limsup_{n\to\infty} \tilde{\mathbb{E}} \, \big\| \phi^1 \big(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \big) \big\|_{L^2(Q)}.
\end{aligned} \tag{3.173}$$

We can therefore substitute (3.173) into (3.172) to get

$$
\begin{aligned}
&\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \left[\operatorname{div} \tilde{\mathbf{u}}_n \, T_k(\tilde{\varrho}_n) - \operatorname{div} \tilde{\mathbf{u}} \, \overline{T_k(\tilde{\varrho})}\right] \,\mathrm{d}x\,\mathrm{d}t \\
&\leq 2 \limsup_{n\to\infty} \left(\tilde{\mathbb{E}} \left\| \phi^2 \operatorname{div} \tilde{\mathbf{u}}_n \right\|_{L^2(Q)} \tilde{\mathbb{E}} \left\| \phi^1 (T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})) \right\|_{L^2(Q)} \right) \\
&\leq 2 \limsup_{n\to\infty} \left(\tilde{\mathbb{E}} \left\| \phi^2 \operatorname{div} \tilde{\mathbf{u}}_n \right\|_{L^2(Q)} \tilde{\mathbb{E}} \left\| \phi^1 (T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})) \right\|_{L^{\gamma+1}(Q)} \right).
\end{aligned} \tag{3.174}
$$

Here, we used the embedding $L^{\gamma+\Theta} \hookrightarrow L^2$ which holds for the choice of $\Theta = 1$. And finally, if we substitute (3.174) into (3.171) and apply Young's inequality, we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1\phi^2 \left| T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \right|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \limsup_{n\to\infty} \left(\tilde{\mathbb{E}} \left\| \phi^2 \operatorname{div} \tilde{\mathbf{u}}_n \right\|_{L^2(Q)} \tilde{\mathbb{E}} \left\| \phi^1 (T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})) \right\|_{L^{\gamma+1}(Q)} \right) \\
&\lesssim \frac{\gamma}{\gamma+1} \sup_n \left(\tilde{\mathbb{E}} \left\| \phi^2 \operatorname{div} \tilde{\mathbf{u}}_n \right\|_{L^2(Q)} \right)^{\frac{\gamma+1}{\gamma}} \\
&+ \frac{1}{\gamma+1} \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_Q \phi^1 \left| T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}) \right|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.175}
$$

The claim thus follow by absolving the last term above into the left-hand side (note that $\frac{1}{\gamma+1} < \frac{2}{5}$ is small enough) and using $(3.106)_5$ keeping in mind that $\frac{\gamma+1}{\gamma} < 2$. $\square$

3.4.16 The renormalized solution for the limit process

With Lemma 3.4.14 in hands, we can now obtain the renormalized continuity equation for the limit process which in turn will finally help us identify the limit of the pressure term. To do this however, we first need to prove the following lemma.

Lemma 3.4.17. Let b be a continuously differentiable function such that $b'(z) = 0$ for all z large enough, say, $z \geq M$. Also for $\phi(x) \in C_c^\infty(\mathbb{R}^3)$, let $\phi\overline{(T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho})) \operatorname{div} \tilde{\mathbf{u}}}$ and $\phi\overline{T_k(\tilde{\varrho})}$ be the limits of the sequences in Lemma 3.4.4 and Lemma 3.4.5 respectively. Then for $Q = (0,T) \times \mathbb{R}^3$, we have that

$$
\phi b'(\overline{T_k(\tilde{\varrho})}) \overline{(T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho})) \operatorname{div} \tilde{\mathbf{u}}} \to 0
$$

in $L^1(\tilde{\Omega} \times Q)$ as $k \to \infty$.

Proof. Let $Q_{k,M} = \Big\{(\omega, t, x) \in \tilde{\Omega} \times Q \,:\, |\overline{T_k(\tilde{\varrho})}| \leq M\Big\}$. Then we obtain

$$\begin{aligned}
&\Big\|\phi b'(\overline{T_k(\tilde{\varrho})})\overline{(T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho}))\operatorname{div}\tilde{\mathbf{u}}}\Big\|_{L^1(\tilde{\Omega}\times Q)} \\
&\leq \left(\sup_{0\leq z\leq M}|b'(z)|\right)\Big\|\phi\overline{(T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho}))\operatorname{div}\tilde{\mathbf{u}}}\Big\|_{L^1(\tilde{\Omega}\times Q)} \\
&\leq \left(\sup_{0\leq z\leq M}|b'(z)|\right)\liminf_{n\to\infty}\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\operatorname{div}\tilde{\mathbf{u}}_n\big\|_{L^1(\tilde{\Omega}\times Q)} \\
&\leq \left(\sup_{0\leq z\leq M}|b'(z)|\right)\left(\sup_n\|\phi\operatorname{div}\tilde{\mathbf{u}}_n\|_{L^2(\tilde{\Omega}\times Q)}\right) \\
&\times\liminf_{n\to\infty}\big\|\phi\left(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\big\|_{L^2(Q_{k,M})} \\
&\lesssim_M \liminf_{n\to\infty}\big\|\phi\left(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\big\|_{L^2(Q_{k,M})}
\end{aligned} \tag{3.176}$$

where we have used weak lower semicontinuity of the norm, Hölder's inequality, the assumption of b and (3.91) combined with proposition 3.3.17.

Using the interpolation inequality, we also have that

$$\begin{aligned}
&\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^2_{L^2(Q_{k,M})} \\
&\leq \big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^{(1-\alpha)(\gamma+1)}_{L^{\gamma+1}(Q_{k,M})}\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^{\alpha}_{L^1(\tilde{\Omega}\times Q))}
\end{aligned} \tag{3.177}$$

for $\alpha = \frac{\gamma-1}{\gamma} \in (0,1)$ and where the above implies

$$\begin{aligned}
&\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|_{L^2(Q_{k,M})} \\
&\leq \big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})}\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^{\frac{\gamma-1}{2\gamma}}_{L^1(\tilde{\Omega}\times Q))}.
\end{aligned} \tag{3.178}$$

And thus,

$$\begin{aligned}
&\liminf_{n\to\infty}\big\|\phi\left(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\big\|_{L^2(Q_{k,M})} \\
&\leq \limsup_{n\to\infty}\big\|\phi\left(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\big\|_{L^2(Q_{k,M})} \\
&\lesssim \limsup_{n\to\infty}\big\|\phi\left(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\big\|^{\frac{\gamma-1}{2\gamma}}_{L^1(Q_{k,M})} \\
&\times\big\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\big\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(\tilde{\Omega}\times Q))}.
\end{aligned} \tag{3.179}$$

However, since

$$\begin{aligned}\left|T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\right| &= \left|\left(T_k(\tilde{\varrho}_n) - (T_k)'_+(\tilde{\varrho}_n)\tilde{\varrho}_n\right)\mathbb{1}_{\{\tilde{\varrho}_n<k\}} + T_k(\tilde{\varrho}_n)\mathbb{1}_{\{k\leq\tilde{\varrho}_n\}}\right| \\ &= \left|T_k(\tilde{\varrho}_n)\,\mathbb{1}_{\{k\leq\tilde{\varrho}_n\}}\right| \leq \left|\tilde{\varrho}_n\,\mathbb{1}_{\{k\leq\tilde{\varrho}_n\}}\right|,\end{aligned} \tag{3.180}$$

it follows from (2.35) and $(3.106)_{4a}$ that

$$\begin{aligned}\limsup_{n\to\infty}\left\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\right\|^{\frac{\gamma-1}{2\gamma}}_{L^1(\tilde{\Omega}\times Q))} &\leq \limsup_{n\to\infty}\left\|\phi\,\tilde{\varrho}_n\,\mathbb{1}_{\{k\leq\tilde{\varrho}_n\}}\right\|^{\frac{\gamma-1}{2\gamma}}_{L^1(\tilde{\Omega}\times Q))} \\ &\leq \left(\frac{1}{k}\right)^{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2}\limsup_{n\to\infty}\left\|\phi\,\tilde{\varrho}_n\right\|^{\frac{\gamma-1}{2\gamma}}_{L^\gamma(\tilde{\Omega}\times Q))} \to 0\end{aligned} \tag{3.181}$$

as $k\to\infty$.

Similarly, we have that

$$\begin{aligned}&\left\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\right\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} = \left\|\phi\,T_k(\tilde{\varrho}_n)\mathbb{1}_{\{k\leq\tilde{\varrho}_n\}}\right\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} \\ &\lesssim \left\|\phi\big(T_k(\tilde{\varrho}_n) - \overline{T_k(\tilde{\varrho})}\big)\right\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} + \left\|\phi\overline{T_k(\tilde{\varrho})}\right\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} \\ &\lesssim \|\phi(\overline{T_k(\tilde{\varrho})} - T_k(\tilde{\varrho}))\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} + \|\phi(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}))\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(\tilde{\Omega}\times Q)} + M^{\frac{\gamma+1}{2\gamma}}\end{aligned} \tag{3.182}$$

where we have applied the triangle inequality followed by the estimate $(a+b)^d \lesssim_d a^d + b^d$ for $a, b \geq 0$, $d > 1$. Now since

$$\begin{aligned}\|\phi(\overline{T_k(\tilde{\varrho})} - T_k(\tilde{\varrho}))\|_{L^{\gamma+1}(Q_{k,M})} &\leq \liminf_{n\to\infty}\|\phi(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}))\|_{L^{\gamma+1}(\tilde{\Omega}\times Q)} \\ &\leq \limsup_{n\to\infty}\|\phi(T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho}))\|_{L^{\gamma+1}(\tilde{\Omega}\times Q)},\end{aligned}$$

and $\frac{1}{2\gamma} < 1$, we obtain from (3.182),

$$\begin{aligned}&\left\|\phi\big(T_k(\tilde{\varrho}_n) - T_k'(\tilde{\varrho}_n)\tilde{\varrho}_n\big)\right\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}(Q_{k,M})} \\ &\qquad\lesssim \limsup_{n\to\infty}\tilde{\mathbb{E}}\int_Q \phi\left|T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})\right|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t + M^{\frac{\gamma+1}{2\gamma}} \\ &\qquad\lesssim \mathbf{osc}_{\gamma+1}[\tilde{\varrho}_n\to\tilde{\varrho}](\tilde{\Omega}\times Q) + M^{\frac{\gamma+1}{2\gamma}} \\ &\qquad\lesssim_M 1\end{aligned} \tag{3.183}$$

for a constant that is independent of k and where we have used Lemma 3.4.14.

Now substituting (3.179), (3.181) and (3.183) into (3.176), we obtain

$$\begin{aligned}
&\left\| \phi b'(\overline{T_k(\tilde{\varrho})}) \overline{(T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho}))\operatorname{div}\tilde{\mathbf{u}}} \right\|_{L^1(\tilde{\Omega}\times Q)} \\
&\qquad \lesssim_{\gamma,M,Q} \left(\frac{1}{k}\right)^{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2} \limsup_{n\to\infty} \left\|\phi\,\tilde{\varrho}_n\right\|_{L^\gamma(\tilde{\Omega}\times Q))}^{\frac{\gamma-1}{2\gamma}} \\
&\qquad \lesssim_{\gamma,M,Q} \left(\frac{1}{k}\right)^{\frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2} \to 0
\end{aligned} \tag{3.184}$$

as $k \to \infty$. □

We can now show the following lemma which is a stochastic version of [43, Section 4.2] and [86, Lemma 7.26]. Similar stochastic versions of this lemma already appears in [16, Section 6] and [100, Section 7] for the treatments on the periodic and bounded spatial domains respectively. We follow the approach of the former with the slight simplification of not requiring the localizing characteristics function introduced in their argument.

Lemma 3.4.18. The renormalized continuity equation

$$\begin{aligned}
\int_{\mathbb{R}^3} b(\tilde{\varrho}(t))\,\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} b(\tilde{\varrho}_0)\,\varphi\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} \Big(& b(\tilde{\varrho})\tilde{\mathbf{u}}\cdot\nabla\varphi \\
& - [\tilde{\varrho}\, b'(\tilde{\varrho}) - b(\tilde{\varrho})]\operatorname{div}\tilde{\mathbf{u}}\,\varphi\Big)\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.185}$$

holds $\tilde{\mathbb{P}}$-a.s. for any $\varphi(x) \in C_c^\infty(\mathbb{R}^3)$ where b satisfies the preamble to (3.20).

Proof. Using (3.146), (3.142), (3.148) and Proposition 3.3.21, we get from (3.141) that $\tilde{\mathbb{P}}$-a.s

$$\partial_t \overline{T_k(\tilde{\varrho})} + \operatorname{div}\left(\overline{T_k(\tilde{\varrho})}\tilde{\mathbf{u}}\right) + \overline{[T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho})]\operatorname{div}\tilde{\mathbf{u}}} = 0 \tag{3.186}$$

in the sense of distributions. The boundedness of $\overline{T_k(\cdot)}$ means that we can apply the regularization result of DiPerna and Lions [30]. So for a continuously differentiable function b with $b'(z) = 0$ when z is large, we get by applying b to (3.186) that

$$\begin{aligned}\partial_t b\left(\overline{T_k(\tilde{\varrho})}\right) + \operatorname{div}\left(b\left(\overline{T_k(\tilde{\varrho})}\right)\tilde{\mathbf{u}}\right) + \left[b'\left(\overline{T_k(\tilde{\varrho})}\right)\overline{T_k(\tilde{\varrho})} - b\left(\overline{T_k(\tilde{\varrho})}\right)\right]\operatorname{div}\tilde{\mathbf{u}} \\ = -b'\left(\overline{T_k(\tilde{\varrho})}\right)\overline{\left[T_k'(\tilde{\varrho})\tilde{\varrho} - T_k(\tilde{\varrho})\right]\operatorname{div}\tilde{\mathbf{u}}}.\end{aligned} \tag{3.187}$$

holds $\tilde{\mathbb{P}}$-a.s. in the sense of distributions.

Now using (2.33), (3.166) and lower semi-continuity of norms, one has that for any $p \in [1, \gamma)$

$$\begin{aligned}\left\|\overline{T_k(\tilde{\varrho})} - \tilde{\varrho}\right\|^p_{L^p(\tilde{\Omega}\times Q)} &\leq \liminf_{n\to\infty} \|T_k(\tilde{\varrho}_n) - \tilde{\varrho}_n\|^p_{L^p(\tilde{\Omega}\times Q)} \\ &\leq \limsup_{n\to\infty} \|T_k(\tilde{\varrho}_n) - \tilde{\varrho}_n\|^p_{L^p(\tilde{\Omega}\times Q)} \\ &\lesssim k^{\frac{1}{\gamma}-\frac{1}{p}} \to 0\end{aligned} \tag{3.188}$$

as $k \to \infty$ since $\frac{1}{\gamma} - \frac{1}{p} < 0$. Thus we have that

$$\overline{T_k(\tilde{\varrho})} \to \tilde{\varrho} \quad \text{in} \quad L^p\left(\tilde{\Omega}\times Q\right) \tag{3.189}$$

for all $p \in [1, \gamma)$.

We can now combine (3.189) with Lemma 3.4.17 to pass to the limit in (3.187). The result then follows. □

3.4.19 Strong convergence of density

To conclude with the identification of the limit density, let us now consider the following function

$$L_k(z) = \begin{cases} z\log(z) & \text{if } z \in [0, k), \\ z\log(k) + z\displaystyle\int_k^z \frac{T_k(s)}{s^2}\,\mathrm{d}s & \text{if } z \in [k, \infty). \end{cases} \tag{3.190}$$

One can easily verify that

$$zL_k'(z) - L_k(z) = T_k(z) \tag{3.191}$$

and that $L_k = b$ satisfies the preamble to (3.20). Substituting (3.190) and (3.191) into (3.185) yields

$$\begin{aligned}\int_{\mathbb{R}^3} L_k(\tilde{\varrho}(t))\,\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} L_k(\tilde{\varrho}_0)\,\varphi\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} \big(L_k(\tilde{\varrho})\tilde{\mathbf{u}}\cdot\nabla\varphi \\ - T_k(\tilde{\varrho})\,\mathrm{div}\,\tilde{\mathbf{u}}\,\varphi\big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.192}$$

$\tilde{\mathbb{P}}$-a.s. and similarly for the sequence (3.141) which becomes

$$\begin{aligned}\int_{\mathbb{R}^3} L_k(\tilde{\varrho}_n(t))\,\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} L_k(\tilde{\varrho}_{0,n})\,\varphi\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} \big(L_k(\tilde{\varrho}_n)\tilde{\mathbf{u}}_n\cdot\nabla\varphi \\ - T_k(\tilde{\varrho}_n)\,\mathrm{div}\,\tilde{\mathbf{u}}_n\,\varphi\big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.193}$$

$\tilde{\mathbb{P}}$-a.s.

Similar to (3.142), we gain by using $(3.106)_4$ and Lemma 2.5.1 that for any $p \in (1,\gamma)$ and any $K \Subset \mathbb{R}^3$,

$$L_k(\tilde{\varrho}_n) \to \overline{L_k(\tilde{\varrho})} \quad \text{in} \quad C_w\big([0,T];L^p(K)\big) \tag{3.194}$$

$$\tilde{\varrho}_n\log(\tilde{\varrho}_n) \to \overline{\tilde{\varrho}\log(\tilde{\varrho})} \quad \text{in} \quad C_w\big([0,T];L^p(K)\big) \tag{3.195}$$

$\tilde{\mathbb{P}}$-a.s. Furthermore since the embedding $C_w\big([0,T];L^p(K)\big) \hookrightarrow C\big([0,T];W^{-1,2}(K)\big)$ is compact, we also have that

$$L_k(\tilde{\varrho}_n) \to \overline{L_k(\tilde{\varrho})} \quad \text{in} \quad C\big([0,T];W^{-1,2}(K)\big) \tag{3.196}$$

$\tilde{\mathbb{P}}$-a.s. for $K \Subset \mathbb{R}^3$.

Also, similar to Lemma 3.4.4, we gain from $(3.106)_8$

$$T_k(\tilde{\varrho}_n)\mathrm{div}\,\tilde{\mathbf{u}}_n \to \overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} \quad \text{in} \quad \big(L^q(0,T;L^q_{\mathrm{loc}}(\mathbb{R}^3)),w\big) \tag{3.197}$$

$\tilde{\mathbb{P}}$-a.s. for some $q > 1$ so that by the use of (3.194), $(3.106)_1$, (3.196), (3.197) and

$(3.106)_5$ we are able to pass to the limit in (3.193) and obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \overline{L_k(\tilde{\varrho}(t))}\,\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} L_k(\tilde{\varrho}_0)\,\varphi\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} \big(\overline{L_k(\tilde{\varrho})}\tilde{\mathbf{u}}\cdot\nabla\varphi \\ - \overline{T_k(\tilde{\varrho})\,\mathrm{div}\,\tilde{\mathbf{u}}}\,\varphi\big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.198}$$

$\tilde{\mathbb{P}}$-a.s.

By taking the difference between (3.192) and (3.198), we then get that

$$\begin{aligned}&\int_{\mathbb{R}^3} \big[\overline{L_k(\tilde{\varrho})} - L_k(\tilde{\varrho})\big](t)\,\varphi\,\mathrm{d}x \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} - T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}\big]\varphi\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad = \int_0^t \int_{\mathbb{R}^3} \big[\overline{L_k(\tilde{\varrho})} - L_k(\tilde{\varrho})\big]\tilde{\mathbf{u}}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{3.199}$$

$\tilde{\mathbb{P}}$-a.s. for any $t \in [0,T]$ and $\varphi(x) \in C_c^\infty(\mathbb{R}^3)$.

To control the right-hand side of (3.199), we now consider the following cut-off function $\varphi = \phi_m$ introduced in [86, Eq. 4.14.12, Eq. 7.11.43]

$$\begin{cases} \phi_m(x) = \phi\Big(\dfrac{x}{m}\Big), \quad m \in \mathbb{N}, \quad \varphi \in C_c^\infty(\mathbb{R}^3), \\ 0 \le \phi(x) \le 1, \quad \phi(x) = \begin{cases} 1, & x \in B_1 \\ 0, & x \in \mathbb{R}^3 \setminus \overline{B_2} \end{cases} \end{cases}$$

whose support, $\mathrm{spt}\,\phi_m \subset B_{2m}$ and which has the property that

$$\sup_{x\in\mathbb{R}^3} \big|\nabla\phi_m(x)\big| \lesssim \frac{1}{m}, \tag{3.200}$$

$$\int_{\mathbb{R}^3} \big|\mathbf{v}\cdot\nabla\phi_m\big|^p \to 0, \quad 2 \le p < 6 \tag{3.201}$$

for any $\mathbf{v} \in D^{1,2}(\mathbb{R}^3)$ as $m \to \infty$. See [86, Exercise 7.66].

Now, since the map $z \mapsto L_k(z)$ is convex, we gain from (3.199),

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} - T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}\big]\phi_m \,\mathrm{d}x\,\mathrm{d}\tau \\
\leq \int_0^t \int_{\mathbb{R}^3} \big[\overline{L_k(\tilde{\varrho})} - L_k(\tilde{\varrho})\big]\tilde{\mathbf{u}}\cdot\nabla\phi_m \,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
\tag{3.202}
$$

$\tilde{\mathbb{P}}$-a.s. for any $t \in [0,T]$ and with the left-hand of (3.202) satisfying

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} - T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}\big]\phi_m \,\mathrm{d}x\,\mathrm{d}\tau \\
= \int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})} - T_k(\tilde{\varrho})\big]\mathrm{div}\,\tilde{\mathbf{u}}\,\phi_m \,\mathrm{d}x\,\mathrm{d}\tau \\
+ \int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} - \overline{T_k(\tilde{\varrho})}\mathrm{div}\,\tilde{\mathbf{u}}\big]\phi_m \,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{3.203}
$$

So that up to the taking of possible subsequences, we gain from (3.171) and semi-continuity,

$$
\begin{aligned}
\tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \big[\overline{T_k(\tilde{\varrho})\mathrm{div}\,\tilde{\mathbf{u}}} - \overline{T_k(\tilde{\varrho})}\mathrm{div}\,\tilde{\mathbf{u}}\big]\phi_m \,\mathrm{d}x\,\mathrm{d}\tau \\
\geq \frac{1}{\lambda+2\nu} \limsup_{n\to\infty} \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^3} \big|T_k(\tilde{\varrho}_n) - T_k(\tilde{\varrho})\big|^{\gamma+1}\phi_m \,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
\tag{3.204}
$$

for any $t \in [0,T]$.

On the other hand, Hölder's inequality, interpolation and the fact that both $\frac{\gamma+1}{2\gamma}$ and $\frac{\gamma-1}{2\gamma}$ are bounded above by one means that up to the taking of possible subsequences, we gain from (3.166)–(3.167),

$$
\begin{aligned}
&\tilde{\mathbb{E}}\int_0^t\int_{\mathbb{R}^3}\big[\overline{T_k(\tilde\varrho)}-T_k(\tilde\varrho)\big]\operatorname{div}\tilde{\mathbf{u}}\,\phi_m\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq \big\|\big(\overline{T_k(\tilde\varrho)}-T_k(\tilde\varrho)\big)\,\phi_m\big\|_{L^2\left(\tilde\Omega\times(0,T)\times\mathbb{R}^3\right)}\big\|\operatorname{div}\tilde{\mathbf{u}}\big\|_{L^2\left(\tilde\Omega\times(0,T)\times\mathbb{R}^3\right)}\\
&\lesssim \big\|\overline{T_k(\tilde\varrho)}-T_k(\tilde\varrho)\big\|^{\frac{\gamma-1}{2\gamma}}_{L^1\left(\tilde\Omega\times(0,T)\times B_{2m}\right)}\big\|\overline{T_k(\tilde\varrho)}-T_k(\tilde\varrho)\big\|^{\frac{\gamma+1}{2\gamma}}_{L^{\gamma+1}\left(\tilde\Omega\times(0,T)\times B_{2m}\right)}\\
&\lesssim \left(\tilde{\mathbb{E}}\int_0^T\int_{B_{2m}}\big|\overline{T_k(\tilde\varrho)}-T_k(\tilde\varrho)\big|\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\times \mathbf{osc}_{\gamma+1}\left[\tilde\varrho_n\to\tilde\varrho\right]\big(\tilde\Omega\times(0,T)\times B_{2m}\big)\\
&\lesssim \tilde{\mathbb{E}}\int_0^T\int_{B_{2m}}\big|\overline{T_k(\tilde\varrho)}-\tilde\varrho\big|\,\mathrm{d}x\,\mathrm{d}t+\tilde{\mathbb{E}}\int_0^T\int_{B_{2m}}\big|T_k(\tilde\varrho)-\tilde\varrho\big|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim_m k^{\frac{1}{\gamma}-1}\to 0
\end{aligned}
\tag{3.205}
$$

as $k\to\infty$ by the use of (2.34)–(2.33).

Now if we let $L^p_2(\mathbb{R}^3)$ be the Orlicz space (refer to Notations at the start of this document), then we gain by a similar estimate as [86, Eq. 7.11.49],

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}\big[\overline{L_k(\tilde\varrho)}-L_k(\tilde\varrho)\big]\tilde{\mathbf{u}}\cdot\nabla\phi_m\,\mathrm{d}x\,\mathrm{d}\tau &\lesssim \big\|\overline{L_k(\tilde\varrho)}-L_k(\tilde\varrho)\big\|_{L^\infty\left(0,T;L^p_2(\mathbb{R}^3)\right)}\\
&\times\big\|\tilde{\mathbf{u}}\cdot\nabla\phi_m\big\|_{L^1\left(0,T;L^2(\mathbb{R}^3)\right)}\big\|\tilde{\mathbf{u}}\cdot\nabla\phi_m\big\|_{L^1\left(0,T;L^{p'}(\mathbb{R}^3)\right)}\\
&\to 0
\end{aligned}
\tag{3.206}
$$

$\tilde{\mathbb{P}}$-a.s. for any $\frac{3}{2}<p\leq\min\{\gamma,2\}$ as $m\to\infty$. Once we observe that $L_k(\tilde\varrho)=c_k\tilde\varrho+b_k(\tilde\varrho)$ where $c_k>0$ and $|b_k(\tilde\varrho)|\leq c(k)$, then we see that the boundedness of the first term on the right-hand side of (3.206) follows from (3.194), $(3.106)_4$ and the fact that weakly continuous functions are essentially bounded in time. The convergence to zero thus follow from (3.200)–(3.201).

By collecting (3.203)–(3.206), we have obtained from (3.202) that for any $K\Subset\mathbb{R}^3$,

$$
\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{\mathbb{E}}\int_0^t\int_K\big|T_k(\tilde\varrho_n)-T_k(\tilde\varrho)\big|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}\tau\leq 0 \tag{3.207}
$$

holds, up to the taking of possible subsequences, for any $t\in[0,T]$ since there exists $m_K>1$ such that if $m>m_K$, then $K\subset B'\subset B_m$ for some ball B'.

Now by the triangle inequality and (2.33)–(2.34), we gain from (3.207),

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^T\int_K\big|\tilde{\varrho}_n-\tilde{\varrho}\big|\,\mathrm{d}x\,\mathrm{d}\tau &= \tilde{\mathbb{E}}\int_0^T\int_K\big|T_k(\tilde{\varrho}_n)-T_k(\tilde{\varrho})\big|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad+\tilde{\mathbb{E}}\int_0^T\int_K\big|T_k(\tilde{\varrho}_n)-\tilde{\varrho}_n\big|\,\mathrm{d}x\,\mathrm{d}\tau+\tilde{\mathbb{E}}\int_0^T\int_K\big|T_k(\tilde{\varrho})-\tilde{\varrho}\big|\,\mathrm{d}x\,\mathrm{d}\tau\\ &\lesssim k^{\frac{1}{\gamma}-1}\end{aligned} \tag{3.208}$$

as $k\to 0$ and thus

$$\tilde{\varrho}_n\to\tilde{\varrho}\quad\text{in}\quad L^1\big(\tilde{\Omega}\times(0,T);L^1_{\mathrm{loc}}(\mathbb{R}^3)\big) \tag{3.209}$$

3.5 Conclusion

Having obtained the strong convergence of density (3.209), we now have all the required results to finally identify our limit solution to (1.16) in the sense of Definition 3.2.6. However, since identifying the limit in the stochastic integral (3.14) appearing in the energy inequality, as well as the Itô correction term (3.14) in (3.12) are a little complicated, we first establish them in the following proposition before stating the aforementioned identification of the limit result.

Proposition 3.5.1. For a.e. $t\in[0,T]$, we have that

$$\int_0^t\int_{\mathbb{R}^3}\tilde{\mathbf{u}}_n\cdot\Phi\big(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)\,\mathrm{d}x\,\mathrm{d}\tilde{W}_n\to\int_0^t\int_{\mathbb{R}^3}\tilde{\mathbf{u}}\cdot\Phi\big(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}}\big)\,\mathrm{d}x\,\mathrm{d}\tilde{W} \tag{3.210}$$

in $L^2(0,T)$ in probability and similarly,

$$\int_0^t\int_{\mathbb{R}^3}\sum_{k\in\mathbb{N}}\frac{1}{2}\frac{|\mathbf{g}_k(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n)|^2}{\tilde{\varrho}_n}\,\mathrm{d}x\,\mathrm{d}s\to\int_0^t\int_{\mathbb{R}^3}\sum_{k\in\mathbb{N}}\frac{1}{2}\frac{|\mathbf{g}_k(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})|^2}{\tilde{\varrho}}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.211}$$

Proof. First of all, we note that $(3.109)_2$ holds for any arbitrary set $K\Subset\mathbb{R}^3$ replacing B_r. Furthermore, we gain from $(3.109)_2$, the $\tilde{\mathbb{P}}$-a.s. uniformly integrability of the family $|\sqrt{\tilde{\varrho}_n}\tilde{\mathbf{u}}_n|^2$. Finally, since $(0,T)\times K$ has finite $\mathcal{L}^4$-Lebesgue measure, it follows from Vitali's convergence theorem that up to the taking of subsequences, the following convergence

$$\sqrt{\tilde{\varrho}_n}\tilde{\mathbf{u}}_n \to \sqrt{\tilde{\varrho}}\tilde{\mathbf{u}} \quad \text{a.e. in} \quad (0,T)\times K$$

holds $\tilde{\mathbb{P}}$-a.s. for any $K \Subset \mathbb{R}^3$ and as such, holds true on the support of the noise (3.1) as well. So in the following, $K \Subset \mathbb{R}^3$ is chosen to coincide with that in (3.1).

Now fix an arbitrary $\kappa > 0$. Then by Egorov's theorem and (3.209), there exists of a measurable set $\mathfrak{O}_\kappa \subset \tilde{\Omega}\times(0,T)\times K$ such that for $\mathfrak{O}_\kappa^c := \big[\tilde{\Omega}\times(0,T)\times K\big]\setminus \mathfrak{O}_\kappa$,

$$(\mathbb{P}\otimes\mathcal{L}^4)(\mathfrak{O}_\kappa^c) < \kappa \tag{3.212}$$

and that

$$\sqrt{\tilde{\varrho}_n}\tilde{\mathbf{u}}_n \to \sqrt{\tilde{\varrho}}\tilde{\mathbf{u}}, \quad \tilde{\varrho}_n \to \tilde{\varrho} \quad \text{uniformly in} \quad \mathfrak{O}_\kappa. \tag{3.213}$$

Given (3.213), we now set

$$\begin{aligned}
\mathfrak{O}_\kappa^1 &:= \big\{(\omega,t,x)\in\mathfrak{O}_\kappa \,:\, \tilde{\varrho}<\kappa\big\},\\
\mathfrak{O}_\kappa^2 &:= \big\{(\omega,t,x)\in\mathfrak{O}_\kappa \,:\, \tilde{\varrho}\geq\kappa\big\}
\end{aligned}$$

so that for a large enough choice of n, we have that

$$\tilde{\varrho}_n < 2\kappa \quad \text{in} \quad \mathfrak{O}_\kappa^1, \quad \tilde{\varrho}_n \geq \frac{\kappa}{2} \quad \text{in} \quad \mathfrak{O}_\kappa^2.$$

Then it follows that for any $t\in[0,T]$ and for any $k\in\mathbb{N}$,

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_0^t\int_{\mathbb{R}^3}\big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\\
&\leq\int_{\mathfrak{O}_\kappa^c}\big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&+\int_{\mathfrak{O}_\kappa^1}\big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&+\int_{\mathfrak{O}_\kappa^2}\big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde{\varrho}_n,\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}.
\end{aligned} \tag{3.214}$$

But from (3.2),

$$
\begin{aligned}
&\int_{\mathfrak{O}_\kappa^c} \big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde\varrho_n,\tilde\varrho_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde\varrho,\tilde\varrho\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\qquad\le \int_{\mathfrak{O}_\kappa^c}\Big(\big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde\varrho_n,\tilde\varrho_n\tilde{\mathbf{u}}_n\big)\big|+\big|\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde\varrho,\tilde\varrho\tilde{\mathbf{u}}\big)\big|\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\qquad\lesssim \int_{\mathfrak{O}_\kappa^c}\Big(\sqrt{\tilde\varrho_n}\sqrt{\tilde\varrho_n}\big|\tilde{\mathbf{u}}_n\big|+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\sqrt{\tilde\varrho}\sqrt{\tilde\varrho}\big|\tilde{\mathbf{u}}\big|+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\qquad\lesssim \int_{\mathfrak{O}_\kappa^c}\Big(\tilde\varrho_n+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}
\end{aligned}
\tag{3.215}
$$

where we have used Young's inequality in the last step above. However, since $\varrho\lesssim 1+\varrho^\gamma$, we gain by Hölder's inequality, $(3.87)_{2,6}$, (3.212) and Proposition 3.3.17,

$$
\begin{aligned}
&\int_{\mathfrak{O}_\kappa^c}\Big(\tilde\varrho_n+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\quad\lesssim \int_{\mathfrak{O}_\kappa^c}\Big(1+\tilde\varrho_n^\gamma+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho^\gamma+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\quad\lesssim \int_{\mathfrak{O}_\kappa^c}\Big(1+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}+\int_{\mathfrak{O}_\kappa^c}\Big(\tilde\varrho_n^\gamma+\tilde\varrho^\gamma\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\quad\lesssim \big[\big(\tilde{\mathbb{P}}\otimes\mathcal{L}^4\big)(\mathfrak{O}_\kappa^c)\big]^{\frac{2\gamma-3}{6\gamma}}\left[\tilde{\mathbb{E}}\int_0^T\int_K\Big(1+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\right]^{\frac{4\gamma+3}{6\gamma}}\\
&\quad+\big[\big(\tilde{\mathbb{P}}\otimes\mathcal{L}^4\big)(\mathfrak{O}_\kappa^c)\big]^{\frac{\gamma+\Theta}{\Theta}}\left[\tilde{\mathbb{E}}\int_0^T\int_K\Big(\tilde\varrho_n^{\gamma+\Theta}+\tilde\varrho^{\gamma+\Theta}\Big)\,\mathrm{d}x\,\mathrm{d}s\right]^{\frac{\gamma}{\gamma+\Theta}}\\
&\quad\lesssim \kappa^q
\end{aligned}
\tag{3.216}
$$

where $q=\max\{\frac{4\gamma+3}{6\gamma},\frac{\gamma}{\gamma+\Theta}\}$. We have therefore shown that

$$
\int_{\mathfrak{O}_\kappa^c} \big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde\varrho_n,\tilde\varrho_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde\varrho,\tilde\varrho\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\lesssim\kappa^q
\tag{3.217}
$$

for some $q>0$. Now using a similar but easier argument as in the above, we gain from the definition of $\mathfrak{O}_k^1$, $(3.87)_1$ and Proposition 3.3.17,

$$
\begin{aligned}
&\int_{\mathfrak{O}_\kappa^1} \big|\tilde{\mathbf{u}}_n\cdot\mathbf{g}_k\big(\tilde\varrho_n,\tilde\varrho_n\tilde{\mathbf{u}}_n\big)-\tilde{\mathbf{u}}\cdot\mathbf{g}_k\big(\tilde\varrho,\tilde\varrho\tilde{\mathbf{u}}\big)\big|\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\qquad\lesssim \int_{\mathfrak{O}_\kappa^1}\Big(\tilde\varrho_n+\tilde\varrho_n\big|\tilde{\mathbf{u}}_n\big|^2+\tilde\varrho+\tilde\varrho\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}}\\
&\qquad\lesssim \kappa\,\tilde{\mathbb{E}}\int_0^T\int_K\Big(1+\big|\tilde{\mathbf{u}}_n\big|^2+\big|\tilde{\mathbf{u}}\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\lesssim\kappa.
\end{aligned}
\tag{3.218}
$$

Notice that on $\mathfrak{O}^2_\kappa$, we have that $\tilde{\mathbf{u}}_n = \frac{\sqrt{\tilde{\varrho}_n}\tilde{\mathbf{u}}_n}{\sqrt{\tilde{\varrho}_n}}$ and $\tilde{\varrho}_n\tilde{\mathbf{u}}_n = \sqrt{\tilde{\varrho}_n}\sqrt{\tilde{\varrho}_n}\tilde{\mathbf{u}}_n$ and thus, we gain from (3.213),

$$\tilde{\mathbf{u}}_n \to \tilde{\mathbf{u}}, \quad \tilde{\varrho}_n\tilde{\mathbf{u}}_n \to \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{a.e. in} \quad \mathfrak{O}^2_\kappa. \tag{3.219}$$

Furthermore by Lemma 3.3.7, in particular $(3.87)_{1,4}$, and Proposition 3.3.17, there exists $q_1, q_2 > 1$ such that

$$\sup_n \|\tilde{\mathbf{u}}_n\|_{L^{q_1}(\mathfrak{O}^2_\kappa)} \leq \sup_n \|\tilde{\mathbf{u}}_n\|_{L^{q_1}(\tilde{\Omega}\times(0,T)\times K)} < \infty$$

and similarly,

$$\sup_n \|\tilde{\varrho}_n\tilde{\mathbf{u}}_n\|_{L^{q_2}(\mathfrak{O}^2_\kappa)} < \infty.$$

That is, both the families $\big(\tilde{\mathbf{u}}_n\big)$ and $\big(\tilde{\varrho}_n\tilde{\mathbf{u}}_n\big)$ are L^q-uniformly integrable on $\mathfrak{O}^2_\kappa$ for some $q > 1$ and thus by Vitali's convergence, we gain that

$$\tilde{\mathbf{u}}_n \to \tilde{\mathbf{u}}, \quad \tilde{\varrho}_n\tilde{\mathbf{u}}_n \to \tilde{\varrho}\tilde{\mathbf{u}} \quad \text{strongly in} \quad L^1\big(\mathfrak{O}^2_\kappa\big). \tag{3.220}$$

Finally we obtain from (3.209) and (3.220),

$$\int_{\mathfrak{O}^2_\kappa} \big|\tilde{\mathbf{u}}_n \cdot \mathbf{g}_k\big(\tilde{\varrho}_n, \tilde{\varrho}_n\tilde{\mathbf{u}}_n\big) - \tilde{\mathbf{u}} \cdot \mathbf{g}_k\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big)\big| \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}\tilde{\mathbb{P}} \to 0 \tag{3.221}$$

as $n \to \infty$.

Since $\kappa > 0$ is arbitrary, by collecting (3.214)–(3.221) and using (3.1), we have shown that $\tilde{\mathbb{P}}$-a.s. for a.e. $t \in [0, T]$,

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_n \cdot \mathbf{g}_k\big(\tilde{\varrho}_n, \tilde{\varrho}_n\tilde{\mathbf{u}}_n\big) \,\mathrm{d}x \to \int_{\mathbb{R}^3} \tilde{\mathbf{u}} \cdot \mathbf{g}_k\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big) \,\mathrm{d}x \tag{3.222}$$

holds for all $k \in \mathbb{N}$.

Now due to the uniform bound (3.40) which still holds on the new probability space, it implies from (3.222) that for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_n \cdot \mathbf{g}_k\big(\tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}_n\big)\,\mathrm{d}x \to \int_{\mathbb{R}^3} \tilde{\mathbf{u}} \cdot \mathbf{g}_k\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big)\,\mathrm{d}x \tag{3.223}$$

in $L^2(0,T)$ $\tilde{\mathbb{P}}$-a.s. and thus combined with the summability of the constants in (3.2)–(3.3), we gain that for a.e. $t \in [0,T]$

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} \big(\tilde{\mathbf{u}}_n \cdot \Phi\big(\tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}_n\big) - \tilde{\mathbf{u}} \cdot \Phi\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big)\big)\,\mathrm{d}x \right\|_{L^2\left(0,t;L_2(\mathfrak{U}_0;\mathbb{R})\right)} \\ &\quad \leq \left(\int_0^T \sum_{k\in\mathbb{N}} \left| \int_{\mathbb{R}^3} \big(\tilde{\mathbf{u}}_n \cdot \mathbf{g}_k\big(\tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}_n\big) - \tilde{\mathbf{u}} \cdot \mathbf{g}_k\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big)\big)\,\mathrm{d}x \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \to 0 \end{aligned} \tag{3.224}$$

$\tilde{\mathbb{P}}$-a.s. as $n \to \infty$ which means that

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_n \cdot \Phi\big(\tilde{\varrho}_n, \tilde{\varrho}_n \tilde{\mathbf{u}}_n\big)\,\mathrm{d}x \to \int_{\mathbb{R}^3} \tilde{\mathbf{u}} \cdot \Phi\big(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}\big)\,\mathrm{d}x \tag{3.225}$$

in $L^2\big(0,t;L_2(\mathfrak{U}_0;\mathbb{R})\big)$ $\tilde{\mathbb{P}}$-a.s.

The claim (3.210) now follow from the application of Lemma 2.4.35 together with $(3.106)_7$.

The proof of (3.211) is similar and in fact easier so we leave that to the reader. □

With this preparation, we can now proof the following result.

Proposition 3.5.2. $\big[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}\big]$ is a finite energy weak martingale solution of (1.16) in the sense of Definition 3.2.6 with initial law Λ.

Proof. Given Lemma 3.3.22, Lemma 3.4.18 and (3.209), it remains to derive the energy inequality.

We first note that, as a consequence of Proposition 3.3.17 (in particular, the equality of laws of the random variables) and Theorem 2.4.31 together with (3.37), we know that $(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \tilde{W}_n)$ satisfies the inequality

$$\begin{aligned}
&\int_{\mathbb{T}^3_{L_n}} \left[\frac{\tilde{\varrho}_n|\tilde{\mathbf{u}}_n|^2}{2} + H\big(\tilde{\varrho}_n, \overline{\varrho}\big)\right](t)\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3_{L_n}} \mathbb{S}(\nabla\tilde{\mathbf{u}}_n) : \nabla\tilde{\mathbf{u}}_n\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \int_{\mathbb{T}^3_{L_n}} \left[\frac{1}{2}\tilde{\varrho}_{0,n}|\tilde{\mathbf{u}}_{0,n}|^2 + H\big(\tilde{\varrho}_{0,n}, \overline{\varrho}\big)\right]\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3_{L_n}} \tilde{\mathbf{u}}_n \cdot \Phi(\tilde{\varrho}_n, \tilde{\varrho}_n\tilde{\mathbf{u}}_n)\mathrm{d}x\,\mathrm{d}\tilde{W}_n \\
&+ \int_0^t \int_{\mathbb{T}^3_{L_n}} \sum_{k\in\mathbb{N}} \frac{1}{2}\frac{|\mathbf{g}_k(\tilde{\varrho}_n, \tilde{\varrho}_n\tilde{\mathbf{u}}_n)|^2}{\tilde{\varrho}_n}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{3.226}$$

$\tilde{\mathbb{P}}$-a.s. for any $t \in [0,T]$ (recall from Proposition 3.3.17 that $n := L_n$ was a subsequence of L).

Now by a similar argument as in the derivation of (3.38), we get that the inequality

$$\int_{\mathbb{T}^3_{L_n}} \left[\frac{1}{2}\tilde{\varrho}_{0,n}|\tilde{\mathbf{u}}_{0,n}|^2 + H\big(\tilde{\varrho}_{0,n}, \overline{\varrho}\big)\right]\mathrm{d}x \leq \int_{\mathbb{R}^3} \left[\frac{1}{2}\tilde{\varrho}_0|\tilde{\mathbf{u}}_0|^2 + H\big(\tilde{\varrho}_0, \overline{\varrho}\big)\right]\mathrm{d}x \tag{3.227}$$

holds $\tilde{\mathbb{P}}$-a.s.

Passage to the limit in the other terms on the right-hand side of (3.226) follows from Proposition 3.5.1 by simply extending the spatial domain to the whole space and then using Proposition 3.5.1.

Now let fix a ball ball $B_r \subset \mathbb{T}^3_{L_n}$. Then following the argument in [42, Page 123], we gain from Lemma 3.3.18 and Hölder's inequality,

$$\begin{aligned}
\|\tilde{\mathbf{m}}(t)\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)} &\leq \|\sqrt{\tilde{\varrho}}(t)\|_{L^{2\gamma}(B_r)}\|\sqrt{\tilde{\varrho}}(t)\tilde{\mathbf{u}}(t)\|_{L^2(B_r)} \\
&= \Big(\|\tilde{\varrho}(t)\|_{L^\gamma(B_r)}\|\tilde{\varrho}(t)|\tilde{\mathbf{u}}(t)|^2\|_{L^1(B_r)}\Big)^{\frac{1}{2}}, \quad t \in [0,T].
\end{aligned} \tag{3.228}$$

Now given that $(3.48)_1$ holds $\tilde{\mathbb{P}}$-a.s. on the new probability space due to Proposition 3.17, we deduce from (3.229) that

$$\|\tilde{\mathbf{m}}(t)\|^2_{L^{\frac{2\gamma}{\gamma+1}}(B_r)} \leq \|\tilde{\varrho}(t)\|_{L^\gamma(B_r)} \sup_{t\in[0,T]} \big\|\tilde{\varrho}(t)|\tilde{\mathbf{u}}(t)|^2\big\|_{L^1(B_r)}, \quad t \in [0,T]. \tag{3.229}$$

Hence $\tilde{\mathbb{P}}$-a.s., $\tilde{\mathbf{m}}(t)$ vanishes a.e. on $\{x \in \mathbb{R}^3 \,:\, \tilde{\varrho}(t) = 0\}$ so that $\tilde{\varrho}^{-1}|\tilde{\mathbf{m}}|^2\mathbb{1}_{\{\tilde{\varrho}(t)>0\}}$ is defined for all $t \in [0,T]$ and equal a.a. on $(0,T)$ to $\varrho(t)|\tilde{\mathbf{u}}(t)|^2$. Physically, this means that the momentum vanishes on the support of any potential vacuum region and as such, the ratio $\tilde{\varrho}^{-1}|\tilde{\mathbf{m}}|^2$ is not singular at zero as one may think.

Moving now, the former, which is convex in $\tilde{\mathbf{m}}$ and $\tilde{\varrho}$ (this can easily be checked by a second derivative test), inherits the regularity of the latter, i.e., $\tilde{\varrho}^{-1}|\tilde{\mathbf{m}}|^2\mathbb{1}_{\{\tilde{\varrho}(t)>0\}} \in L^\infty(0,T;L^1(B_r))$ $\tilde{\mathbb{P}}$-a.s. The aforementioned regularity is a consequence of $(3.87)_2$ and Proposition 3.3.17. Hence we gain from (3.106) and lower semicontinuity of convex functions, see for instant [42, Theorem 10.20],

$$\begin{aligned}\liminf_{n\to\infty}\int_{\mathbb{T}^3_{L_n}}&\left[\frac{1}{2}\frac{|\tilde{\mathbf{m}}_n|^2}{\tilde{\varrho}_n}+H\big(\tilde{\varrho}_n,\overline{\varrho}\big)\right](t)\,\mathrm{d}x\\&\geq\liminf_{n\to\infty}\int_{\mathbb{R}^3}\chi_{B_r}\left[\frac{1}{2}\frac{|\tilde{\mathbf{m}}_n|^2}{\tilde{\varrho}_n}+H\big(\tilde{\varrho}_n,\overline{\varrho}\big)\right](t)\,\mathrm{d}x\\&\geq\int_{\mathbb{R}^3}\chi_{B_r}\left[\frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}}+H\big(\tilde{\varrho},\overline{\varrho}\big)\right](t)\,\mathrm{d}x.\end{aligned}\tag{3.230}$$

Now since $H\big(\tilde{\varrho},\overline{\varrho}\big)$ is a convex function in $\tilde{\varrho}$ with a minimum value of zero at $\overline{\varrho}>0$, it implies that for any $t\in[0,T]$, the function

$$f_r(t):=\chi_{B_r}\left[\frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\varrho}+H\big(\tilde{\varrho},\overline{\varrho}\big)\right](t)$$

is nonnegative. Furthermore, it converges a.e. as $r\to\infty$ in $\mathbb{R}^3$ to

$$\left[\frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\varrho}+H\big(\tilde{\varrho},\overline{\varrho}\big)\right](t).$$

So it follows that,

$$\lim_{r\to\infty}\int_{\mathbb{R}^3}\chi_{B_r}\left[\frac{\tilde{\varrho}|\tilde{\mathbf{u}}|^2}{2}+H\big(\tilde{\varrho},\overline{\varrho}\big)\right](t)\,\mathrm{d}x=\int_{\mathbb{R}^3}\left[\frac{\tilde{\varrho}|\tilde{\mathbf{u}}|^2}{2}+H\big(\tilde{\varrho},\overline{\varrho}\big)\right](t)\,\mathrm{d}x.\tag{3.231}$$

A similar argument as (3.230)–(3.231) also yields

$$\lim_{n\to\infty}\int_0^t\int_{\mathbb{T}^3_{L_n}}\mathbb{S}(\nabla\tilde{\mathbf{u}}_n):\nabla\tilde{\mathbf{u}}_n\,\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_{\mathbb{R}^3}\mathbb{S}(\nabla\tilde{\mathbf{u}}):\nabla\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}s\tag{3.232}$$

$\tilde{\mathbb{P}}$-a.s. for any $t\in[0,T]$ (recall (1.10)).

By collecting the various information above, we finally obtain the energy inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{\tilde{\varrho}|\tilde{\mathbf{u}}|^2}{2} + H\big(\tilde{\varrho}, \overline{\varrho}\big) \right](t) \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla \tilde{\mathbf{u}}) : \nabla \tilde{\mathbf{u}} \,\mathrm{d}x \,\mathrm{d}s \\
&\leq \int_{\mathbb{R}^3} \left[\frac{1}{2} \tilde{\varrho}_0 |\tilde{\mathbf{u}}_0|^2 + H\big(\tilde{\varrho}_0, \overline{\varrho}\big) \right] \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \tilde{\mathbf{u}} \cdot \Phi(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}) \,\mathrm{d}x \,\mathrm{d}\tilde{W} \\
&+ \int_0^t \int_{\mathbb{R}^3} \sum_{k\in\mathbb{N}} \frac{1}{2} \tilde{\varrho}^{-1} |\mathbf{g}_k(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}})|^2 \,\mathrm{d}x \,\mathrm{d}s
\end{aligned} \tag{3.233}$$

$\tilde{\mathbb{P}}$-a.s. for any $t \in [0, T]$.

This finishes the proof. □

3.6 Relative energy inequality

At this point, we have completed the proof of our main theorem for this chapter. However, we can show that the notion of a solution $(\varrho, \mathbf{u})$ to (1.16) that we have constructed enjoys an additional useful property. More precisely, we show that any such solution would satisfy a *relative energy inequality.* This relative energy inequality is a tool which enables us to compare $(\varrho, \mathbf{u})$ with some smooth comparison functions. It is a consequence of the energy inequality (3.12) and its proof relies on several applications of Itô's formula in infinite dimensions. The relative energy inequality for the stochastic compressible Navier–Stokes system was first derived in [12] for periodic boundary conditions and we can show that the same ideas apply on the whole space $\mathbb{R}^3$.

To proceed, let recall that the energy inequality is given equivalently by (3.12)

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho, \overline{\varrho}) \right] \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \,\mathrm{d}x \,\mathrm{d}s \\
&\quad \leq \int_{\mathbb{R}^3} \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0, \overline{\varrho}) \right] \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \mathbf{u} \cdot \Phi(\varrho, \varrho\mathbf{u}) \,\mathrm{d}x \,\mathrm{d}W \\
&\quad + \frac{1}{2} \int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \frac{|\mathbf{g}_k(\varrho, \varrho\mathbf{u})|^2}{\varrho} \,\mathrm{d}x \,\mathrm{d}s
\end{aligned} \tag{3.234}$$

$\mathbb{P}$-a.s. for any $t \in [0, T]$.

3.6.1 General framework of comparison functions

Our aim now is to 'measure the distance' between $(\varrho, \mathbf{u})$ satisfying (3.234) and a pair of *test functions* $\big(r\,,\,\mathbf{U}\big)$ solving a system of SPDE. As will be shown soon, the later is expected to have sufficient regularity and hence, its classification as a *test functions*.

To proceed with the details, we let $\big(r\,,\,\mathbf{U}\big)$ be a pair of stochastic processes which are adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Similar to deterministic problems, $\big(r\,,\,\mathbf{U}\big)$ being a test function means that we expect it to behave like $(\varrho, \mathbf{u})$ and thus, we assume that the former satisfy the system

$$\begin{aligned} \mathrm{d}r &= D^d_t r\,\mathrm{d}t + \mathbb{D}^s_t r\,\mathrm{d}W, \\ \mathrm{d}\mathbf{U} &= D^d_t \mathbf{U}\,\mathrm{d}t + \mathbb{D}^s_t \mathbf{U}\,\mathrm{d}W. \end{aligned} \tag{3.235}$$

In the above, $D^d_t r$ and $D^d_t \mathbf{U}$ are smooth functions of (ω, t, x) whereas $\mathbb{D}^s_t r$ and $\mathbb{D}^s_t \mathbf{U}$ belongs to the function space $L_2\big(\mathfrak{U}; L^2(\mathbb{R}^3)\big)$ for a.e $(\omega, t) \in \Omega \times [0, T]$. Specifically, by a smooth approximation in Theorem 2.4.37, we assume that

$$(r - \overline{\varrho}) \in C^\infty_c\big([0, T] \times \mathbb{R}^3\big), \tag{3.236}$$

$$\mathbf{U} \in C^\infty_c\big([0, T] \times \mathbb{R}^3\big) \tag{3.237}$$

$\mathbb{P}$-a.s. and for all $1 \leq q < \infty$,

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} \|r\|^2_{W^{1,q}(\mathbb{R}^3)}\bigg]^q + \mathbb{E}\bigg[\sup_{t\in[0,T]} \|\mathbf{U}\|^2_{W^{1,q}(\mathbb{R}^3)}\bigg]^q \leq c(q),$$

$$0 < \underline{r} \leq r(t, x) \leq \overline{r} \quad \mathbb{P}\text{-a.s.} \tag{3.238}$$

Moreover, r and $\mathbf{U}$ satisfy

$$\begin{aligned} &D^d_t r, D^d_t \mathbf{U} \in L^q\big(\Omega; L^q(0, T; W^{1,q}(\mathbb{R}^3))\big), \\ &\mathbb{D}^s_t r, \mathbb{D}^s_t \mathbf{U} \in L^2\big(\Omega; L^2\big(0, T; L_2(\mathfrak{U}; L^2(\mathbb{R}^3))\big)\big), \end{aligned}$$

as well as

$$\begin{aligned} \Big(\sum_{k\in\mathbb{N}} |\mathbb{D}_t^s r(e_k)|^q \Big)^{\frac{1}{q}} &\in L^q(\Omega; L^q(0,T; L^q(\mathbb{R}^3))), \\ \Big(\sum_{k\in\mathbb{N}} |\mathbb{D}_t^s \mathbf{U}(e_k)|^q \Big)^{\frac{1}{q}} &\in L^q(\Omega; L^q(0,T; L^q(\mathbb{R}^3))). \end{aligned} \tag{3.239}$$

With this preamble, we can now construct the relative energy inequality. Before proceeding however, we emphasis that for the following computations to be made rigorous, a preliminary step explained in Remark 3.4.9 is required.

Firstly, we consider the function

$$f^1(\varrho, \mathbf{U}) = \int \frac{1}{2}\varrho |\mathbf{U}|^2 \,\mathrm{d}x \tag{3.240}$$

which may be considered as the *relative kinetic energy* between the system (1.16) and (3.235). Notice that given the regularity of the density ϱ and (3.237), the function (3.240) is well defined for a.e. $(\omega, t) \in \Omega \times [0,T]$.

From the continuity equation $(1.16)_1$ and integration by parts, it follows that for any $t \in [0,T]$,

$$\begin{aligned} \int_0^t f^1_\varrho \,\mathrm{d}\varrho &= -\int_0^t \int_{\mathbb{R}^3} \frac{1}{2}|\mathbf{U}|^2 \mathrm{div}(\varrho \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \mathbf{U} \cdot \mathbf{U} \,\mathrm{d}x \,\mathrm{d}s. \end{aligned} \tag{3.241}$$

A similar argument for the same function using (3.235) yields

$$\int_0^t f^1_{\mathbf{U}} \,\mathrm{d}\mathbf{U} = \int_0^t \int_{\mathbb{R}^3} \varrho \mathbf{U} \cdot D_t^d \mathbf{U} \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^3} \varrho \mathbf{U} \cdot \mathbb{D}_t^s \mathbf{U} \,\mathrm{d}x \,\mathrm{d}W. \tag{3.242}$$

Finally since the second derivative of f^1 in the ϱ component vanishes, it remains to compute $f^1_{\mathbf{U}\mathbf{U}}$. This yields

$$\frac{1}{2}\int_0^t f^1_{\mathbf{U}\mathbf{U}} \,\mathrm{d}\langle\langle \mathbf{U} \rangle\rangle = \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \varrho \big| \mathbb{D}_t^s \mathbf{U}(e_k) \big|^2 \,\mathrm{d}x \,\mathrm{d}s. \tag{3.243}$$

Collecting (3.241)–(3.243), it follows from Itô's formula that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{1}{2}\varrho|\mathbf{U}|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^3} \frac{1}{2}\frac{\big|\big(\varrho\mathbf{U}\big)(0)\big|^2}{\varrho(0)} \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\nabla\mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{U}\cdot D_t^d\mathbf{U}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{U}\cdot \mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\,\mathrm{d}W \\
&+ \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \varrho\big|\mathbb{D}_t^s\mathbf{U}(e_k)\big|^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.244}$$

A similar application of Itô's formula to the function $f^2(\mathbf{m},\mathbf{U}) = \int \mathbf{m}\mathbf{U}\,\mathrm{d}x$ where $\mathbf{m} = \varrho\mathbf{u}$ is the momentum yields

$$\begin{aligned}
\int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x &= \int_{\mathbb{R}^3} (\varrho\mathbf{u})(0)\cdot\mathbf{U}(0)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot D_t^d\mathbf{U}\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\,\mathrm{d}W - \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{U}\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} p(\varrho)\,\mathrm{div}(\mathbf{U})\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\nabla\mathbf{U}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \mathbf{U}\cdot\Phi(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}W \\
&+ \int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \mathbb{D}_t^s\mathbf{U}(e_k)\cdot\mathbf{g}_k(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.245}$$

Note that (3.245) utilises the momentum balance equation $(1.16)_2$ and (3.235).

Now using the identity $rP'(r) - P(r) = p(r)$ (recall (3.17)), we get by application of Itô's formula to $f^3(r) = \int p(r)\,\mathrm{d}x$ where $p(r) = ar^\gamma$ is the pressure, that

$$\begin{aligned}
\int_{\mathbb{R}^3} \big[rP'(r) - P(r)\big](t)\,\mathrm{d}x &= \int_{\mathbb{R}^3} \big[rP'(r) - P(r)\big](0)\,\mathrm{d}x \\
&+ \int_0^t \int_{\mathbb{R}^3} p'(r)\,D_t^d r\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^3} p'(r)\,\mathbb{D}_t^s r\,\mathrm{d}x\,\mathrm{d}W \\
&+ \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} p''(r)\big|\mathbb{D}_t^s r(e_k)\big|^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.246}$$

Final application of Itô's formula to $f^4(\varrho, r) = \int \varrho\, P'(r)\,\mathrm{d}x$ yields

$$\begin{aligned}
\int_{\mathbb{R}^3} \varrho P'(r)\,\mathrm{d}x &= \int_{\mathbb{R}^3} \varrho(0) P'(r(0))\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \varrho \nabla P'(r) \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \varrho P''(r) \cdot D_t^d r\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^3} \varrho P''(r) \cdot \mathbb{D}_t^s r\,\mathrm{d}x\,\mathrm{d}W \\
&+ \frac{1}{2} \int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \varrho P'''(r) \big|\mathbb{D}_t^s r(e_k)\big|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.247}$$

Now since the following identities

$$\begin{aligned}
\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 &= \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\mathbf{U}|^2 - \varrho\mathbf{u}\cdot\mathbf{U} \\
H(\varrho, r) &= P(\varrho) - P'(r)(\varrho - r) - P(r) \\
&= P(\varrho) - P'(r)\varrho + [P'(r)r - P(r)]
\end{aligned}$$

hold, for the *relative energy functional*

$$\mathcal{E}\left(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\right) = \int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho, r)\right]\mathrm{d}x,$$

we gain by collecting (3.234) and (3.244)–(3.247) that for any $t \in [0, T]$,

$$\begin{aligned}
&\mathcal{E}\big(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\big)(t) + \int_0^t \int_{\mathbb{R}^3} \big[\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla\mathbf{U})\big] : (\nabla\mathbf{u} - \nabla\mathbf{U})\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \mathcal{E}\big(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\big)(0) + M_{RE}(t) + \int_0^t \mathcal{R}\big(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\big)(s)\,\mathrm{d}s
\end{aligned} \tag{3.248}$$

holds $\mathbb{P}$-a.s. where

$$\begin{aligned}
\mathcal{R}\big(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\big) &= \int_{\mathbb{R}^3} \mathbb{S}(\nabla\mathbf{U}) : (\nabla\mathbf{U} - \nabla\mathbf{u})\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \varrho\big(D_t^d\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U}\big)\cdot\big(\mathbf{U} - \mathbf{u}\big)\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \big[(r-\varrho)P''(r)D_t^d r + \nabla P'(r)\cdot(r\mathbf{U} - \varrho\mathbf{u})\big]\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \big[p(r) - p(\varrho)\big]\mathrm{div}(\mathbf{U})\,\mathrm{d}x \\
&+ \frac{1}{2}\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \varrho \left|\frac{\mathbf{g}_k(\varrho, \varrho\mathbf{u})}{\varrho} - \mathbb{D}_t^s\mathbf{U}(e_k)\right|^2\mathrm{d}x \\
&+ \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \big[p''(r) - \varrho P'''(r)\big]\big|\mathbb{D}_t^s r(e_k)\big|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.249}$$

and M_{RE} is a real valued square integrable martingale given by

$$\begin{aligned} M_{RE}(t) &= \int_0^t \int_{\mathbb{R}^3} \big(\mathbf{u} - \mathbf{U}\big) \cdot \Phi(\varrho, \varrho\mathbf{u}) \, \mathrm{d}x \, \mathrm{d}W \\ &- \int_0^t \int_{\mathbb{R}^3} \varrho(\mathbf{u} - \mathbf{U}) \cdot \mathbb{D}_t^s \mathbf{U} \, \mathrm{d}x \, \mathrm{d}W + \int_0^t \int_{\mathbb{R}^3} \big(p'(r) - \varrho P''(r)\big) \mathbb{D}_t^s r \, \mathrm{d}x \, \mathrm{d}W. \end{aligned} \tag{3.250}$$

Remark 3.6.2. Notice that in (3.249), we have used the following identity:

$$\iint \big[\nabla P'(r) \cdot r\mathbf{U} + p(r) \, \mathrm{div}\mathbf{U}\big] \, \mathrm{d}x \, \mathrm{d}s = 0. \tag{3.251}$$

This can be verified by integrating by parts and keeping note of the identity $rP'(r) - P(r) = p(r)$.

Chapter 4

The low Mach number limit result

4.1 Introduction

A fundamental question in compressible fluid mechanics is the relation to the incompressible model. If the Mach number - which is a dimensionless quantity representing the ratio of a characteristic velocity in the flow to the speed of sound in the fluid - is small, the fluid should behave asymptotically like an incompressible one, provided velocity and viscosity are small, and we are looking at large time scales, see [63]. Indeed, in this small Mach number regime, the time scale is inversely proportional to the Mach number, i.e., of order $\frac{1}{\varepsilon}$ where $\mathrm{Ma} = \varepsilon \in (0,1]$. Thus if we make the change of variables $\varrho_\varepsilon(t,x) = \varrho\left(\frac{t}{\varepsilon}, x\right)$, $\mathbf{u}_\varepsilon(t,x) = \frac{1}{\varepsilon}\mathbf{u}\left(\frac{t}{\varepsilon}, x\right)$, $\nu(\varepsilon) = \varepsilon\nu$, $\lambda(\varepsilon) = \varepsilon\lambda$ and $\mathbf{f}(\varepsilon) = \varepsilon^2\mathbf{f}$, then the system of equation (1.11) becomes

$$\begin{aligned}
\partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) &= 0, \\
\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla p(\varrho_\varepsilon) &= \operatorname{div}\big[\mathbb{S}(\nabla \mathbf{u}_\varepsilon)\big] + \varrho_\varepsilon \mathbf{f}, \qquad (4.1) \\
\operatorname{div}\big[\mathbb{S}(\nabla \mathbf{u}_\varepsilon)\big] &= \nu \Delta \mathbf{u}_\varepsilon + (\lambda + \nu)\nabla \operatorname{div}(\mathbf{u}_\varepsilon),
\end{aligned}$$

see [73]. This change of variables leading to (4.1) can be recast as

$$t \to \varepsilon t, \qquad x \to x, \quad \mathbf{u} \to \varepsilon \mathbf{u}, \qquad \nu \to \varepsilon\nu, \quad \lambda \to \varepsilon\lambda.$$

However, this set of rescaling is not unique and one can use instead the following set of scaling

$$
\begin{aligned}
&t \to \varepsilon^2 t, && x \to \varepsilon x, \quad \mathbf{u} \to \varepsilon \mathbf{u}, && \nu \to \nu, \quad \lambda \to \lambda,\\
&t \to t, && x \to x/\varepsilon, \quad \mathbf{u} \to \varepsilon \mathbf{u}, && \nu \to \varepsilon^2 \nu, \quad \lambda \to \varepsilon^2 \lambda.
\end{aligned}
$$

For more on these change of variables, see [2, Eq. 1.2] and the references therein. For the sake of total clarity, it is worth mentioning that the extra parameters mentioned in [2, Eq. 1.2], i.e., the Reynolds number $1/\mu$ and the Péclet number κ, are not relevant in this section but proofs useful when treating for instant, inviscid limit results and/or analysing heat conducting fluids.

As ε approaches zero in (4.1), the resultant equation is expected to model fluids which are incompressible. The problem has been studied rigorously in the deterministic case in for example [73, 74, 76, 24], as a singular limit problem. A major problem to overcome is the rapid oscillation of acoustic waves due to the lack of compactness. A stochastic counterpart of this theory has very recently been established in [11]. The limit as $\varepsilon \to 0$ of the system

$$
\begin{aligned}
\mathrm{d}\varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon)\mathrm{d}t &= 0,\\
\mathrm{d}(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \Big[\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \operatorname{div}(\mathbb{S}(\nabla \mathbf{u}_\varepsilon)) + \frac{1}{\varepsilon^2}\nabla \varrho_\varepsilon^\gamma\Big]\mathrm{d}t &= \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\mathrm{d}W_\varepsilon,\\
\mathbb{S}(\nabla \mathbf{u}_\varepsilon) = \nu\left(\nabla \mathbf{u}_\varepsilon + \nabla^T \mathbf{u}_\varepsilon\right) + \lambda \operatorname{div}(\mathbf{u}_\varepsilon \mathbb{I}), \quad \nu > 0, \lambda &+ \frac{2}{3}\nu \geq 0, \gamma > \frac{3}{2},
\end{aligned}
\tag{4.2}
$$

has been analysed under periodic boundary conditions. Given a sequence of *finite energy weak martingale solution* of (4.2) in the sense of Definition 3.2.6 where $\varepsilon \in (0,1)$, its limit (as $\varepsilon \to 0$) is indeed a *weak martingale solution* in the sense of Definition 4.2.7 below to the following incompressible system:

$$
\begin{aligned}
\operatorname{div}(\mathbf{U}) &= 0,\\
\mathrm{d}(\mathbf{U}) + [\operatorname{div}(\mathbf{U} \otimes \mathbf{U}) - \nu \Delta \mathbf{U} + \nabla \pi]\mathrm{d}t &= \mathcal{P}\Phi(1, \mathbf{U})\mathrm{d}W.
\end{aligned}
\tag{4.3}
$$

Here π is the associated pressure and $\mathcal{P}$ is the Helmholtz projection onto the space of solenoidal vector fields.

A major drawback in the approach in [11] is that the noise coefficient $\Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)$ has to be linear in the momentum $\varrho_\varepsilon \mathbf{u}_\varepsilon$. This is due to the aforementioned lack of compactness of momentum when ε passes to zero. This cannot even be improved in the deterministic case. The situation on the whole space however, is much better as a consequence of dispersive estimates for the acoustic wave equations, see Proposition 4.7.6. We apply them to the stochastic wave equation and hence are able to prove strong convergence of the momentum, see Lemma 4.7.11. Based on this, we are able to prove the convergence of (4.2) to (4.3) under much more general assumptions on the noise coefficients. See Theorem 4.2.9 for details.

4.2 Preliminaries

4.2.1 Mild and weak solutions

We summarize in this section, a stochastic analytic tool pertinent to our subsequent analysis. This is taken from [21] where we change notations, wherever possible, to conform to our settings.

We fix a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and let $W(t)$ be an $(\mathscr{F}_t)$-cylindrical Wiener process.

We consider the problem

$$\begin{aligned} \mathrm{d}x(t) &= \big[\mathcal{A}x(t) + f(t)\big]\,\mathrm{d}t + \Phi(x(t))\,\mathrm{d}W, \\ x(0) &= x_0 \end{aligned} \tag{4.4}$$

defined on a time interval $[0,T]$, where $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $S(\cdot)$ in H, x_0 is an $\mathscr{F}_0$-measurable H-valued random variable, f is a predictable process with local integrable trajectories and finally Φ belongs to

$$\Phi \in \mathcal{N}_W^2 := \mathcal{N}_W^2\big(0, T; L_2(\mathfrak{U}_0; H)\big). \tag{4.5}$$

That is, Φ is $L_2(\mathfrak{U}_0; H)$-predictable[1] such that

$$\|\Phi\|_T = \left[\mathbb{E}\int_0^T \|\Phi(s)\|^2_{L_2(\mathfrak{U}_0;H)}\,\mathrm{d}s\right]^{\frac{1}{2}} < \infty.$$

With this preparation, we now state and compare two definitions of a solution to (4.4).

To do this, we first denote by $\mathcal{A}^*$, the adjoint of the operator $\mathcal{A}$. Then firstly, the following definition gives the stochastic analogue of a distributional or weak solution in deterministic PDEs.

Definition 4.2.2. An H-valued predictable process $x(t)$, $t \in [0, T]$ is a *weak solution* of (4.4) if $\mathbb{P}$-a.s., the integral equation

$$\begin{aligned}\big\langle x(t), \varphi\big\rangle = \big\langle x_0, \varphi\big\rangle + \int_0^t \big\langle x(s), \mathcal{A}^*\varphi\big\rangle\,\mathrm{d}s + \int_0^t \big\langle f(s), \varphi\big\rangle\,\mathrm{d}s \\ + \int_0^t \big\langle \Phi\big(x(s)\big)\,\mathrm{d}W(s), \varphi\big\rangle\end{aligned}$$

holds and is well-defined for any $t \in [0, T]$ and $\varphi \in \mathrm{Dom}(\mathcal{A}^*)$.

Remark 4.2.3. It is enough for Definition 4.2.2 to hold for φ belonging to a dense subset of $\mathrm{Dom}(\mathcal{A}^*)$.

Next, Definition 4.2.4 can be interpreted as a stochastic Duhamel's formula.

Definition 4.2.4. An H-valued predictable process $x(t)$, $t \in [0, T]$ is a *mild solution* of (4.4) if $\mathbb{P}$-a.s., the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s + \int_0^t S(t-s)\Phi\big(x(s)\big)\,\mathrm{d}W(s)$$

[1] that is, $\Phi : [0, T] \times \Omega \to L_2(\mathfrak{U}_0; H)$ is $\mathcal{P}_T/\mathcal{B}(L_2(\mathfrak{U}_0; H))$ measurable and where $\mathcal{P}_T$ is the σ-algebra $\sigma(Y : [0, T] \times \Omega \to \mathbb{R})$ for right-continuous function Y that is adapted to the filtration $\mathscr{F}_t$, $t \in [0, T]$.

holds and is well-defined for any $t \in [0,T]$.

Given the above two definitions, we can conclude with the following theorem which is a partial result of [21, Theorem 6.5].

Theorem 4.2.5. *Assume that* $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset H \to H$ *is an infinitesimal generator of a strongly continuous semigroup* $S(\cdot)$ *in* H *and let* $\Phi \in \mathcal{N}_W^2$*. Then a weak solution is always a mild solution of* (4.4).

4.2.6 Notion of a solution for the limit system

In this chapter, we are concerned with constructing a distributional solution as the limit to any family of *finite energy weak martingale solution* of (4.1). Before we do that, let first clarify a minor point about the definition of a solution to (4.1). For the avoidance of doubt, we remark that we have replaced the second equation in (3.11) by

$$\begin{aligned}\langle \varrho\mathbf{u}(t),\phi\rangle &= \langle \varrho\mathbf{u}(0),\phi\rangle + \int_0^t \langle \varrho\mathbf{u}\otimes\mathbf{u},\nabla\phi\rangle \mathrm{d}s - \nu\int_0^t \langle \nabla\mathbf{u},\nabla\phi\rangle \mathrm{d}s\\ &\quad - (\lambda+\nu)\int_0^t \langle \mathrm{div}\mathbf{u},\mathrm{div}\phi\rangle \mathrm{d}s + \frac{1}{\varepsilon^2}\int_0^t \langle \varrho^\gamma,\mathrm{div}\phi\rangle \mathrm{d}s\\ &\quad + \int_0^t \langle \Phi(\varrho,\varrho\mathbf{u})\mathrm{d}W,\phi\rangle,\end{aligned} \tag{4.6}$$

in this section. This corresponds to the choice of pressure coefficient $a = \varepsilon^{-2}$, the quantity we are referring to as the squared-reciprocal of the Mach number. A similar remark holds for the energy estimate (3.12) and the pressure potential (3.13).

We also remark that the second item in Definition 3.2.6 is not required in this section. As such, a reader may ignore it in the definition of a *finite energy weak martingale solution.*

We now give the precise definition of a solution to the limit system (4.3).

Definition 4.2.7. If Λ is a Borel probability measure on $L^2_{div}(\mathbb{R}^3)$, then we say that $[(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P}),\mathbf{u},W]$ is a *weak martingale solution* of Eq. (4.3) with initial law Λ provided:

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,

2. W is a $(\mathscr{F}_t)$-cylindrical Wiener process,

3. $\mathbf{u}$ is $(\mathscr{F}_t)$-adapted, $\mathbf{u} \in C_w\left([0,T]; L^2_{\mathrm{div}}(\mathbb{R}^3)\right) \cap L^2(0,T; W^{1,2}_{\mathrm{div}}(\mathbb{R}^3))$ $\mathbb{P}$-a.s. and,
$$\mathbb{E}\left[\sup_{(0,T)} \|\mathbf{u}\|^2_{L^2(\mathbb{R}^3)}\right]^p + \mathbb{E}\left[\left(\int_0^T \|\mathbf{u}\|^p_{W^{1,2}(\mathbb{R}^3)} \mathrm{d}t\right)^p\right] < \infty \text{ for all } 1 \leq p < \infty,$$

4. $\Lambda = \mathbb{P} \circ (\mathbf{u}(0))^{-1}$,

5. for all $\phi \in C^\infty_{c,\mathrm{div}}(\mathbb{R}^3)$ and all $t \in [0,T]$, it holds $\mathbb{P}$-a.s.
$$\langle \mathbf{u}(t), \phi\rangle = \langle \mathbf{u}(0), \phi\rangle + \int_0^t \left[\langle \mathbf{u} \otimes \mathbf{u}, \nabla\phi\rangle - \nu\langle \nabla\mathbf{u}, \nabla\phi\rangle\right] \mathrm{d}s + \int_0^t \langle \mathcal{P}\Phi(1, \mathbf{u})\mathrm{d}W, \phi\rangle,$$

Existence of weak martingale solutions as defined in Definition 4.2.7 has been shown to exist under suitable growth conditions on the noise term. We refer the reader to [82], albeit stated in the Stratonovich sense. A whole Euclidean space existence result stated in the Itô form appears to be absent from the literatures although it is certainly expected. However, this is a by product of the singular limit problem that we study in this chapter. See Theorem 4.2.9 below. For bounded domains, see for example, [18, 49].

4.2.8 Main Theorem

This section contains a result from [80, Theorem 2]. We improve results in [11] where the second or momentum term in (1.17) was only a linear function of momentum and the analysis was done on the 3-torus rather than on the whole space $\mathbb{R}^3$.

Working on the torus meant by taking smooth test functions (not necessarily having compact support), they avoided the usual boundary problem with integrating by parts. However, they pay the price by being unable to apply dispersive estimates which exists when working on the whole space. These estimates follow when one works in frequency space by applying Fourier transforms to the macroscopic state variables.

Below is the main result.

Theorem 4.2.9. *Let Λ be a given Borel probability measure on $L^2(\mathbb{R}^3)$. For $\varepsilon \in (0,1)$ and $\gamma > 3/2$, we let Λ_ε be a family of Borel probability measures on $L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)$ such that*

$$\Lambda_\varepsilon\Big\{(\varrho, \mathbf{m}) \in L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3) \,:\, |\varrho - 1| \leq \varepsilon M\Big\} = 1 \tag{4.7}$$

holds for a constant $M > 0$ which is independent of $\varepsilon \in (0,1)$.

Also for all $p \in [1, \infty)$, we assume that the following moment estimate

$$\int_{L^\gamma_x \times L^{\frac{2\gamma}{\gamma+1}}_x} \left\| \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + H(\varrho) \right\|^p_{L^1_x} \mathrm{d}\Lambda_\varepsilon(\varrho, \mathbf{m}) \leq c_p < \infty, \tag{4.8}$$

holds uniformly in ε where

$$H(\varrho) = \frac{1}{\varepsilon^2(\gamma - 1)}\left[\varrho^\gamma - 1 - \gamma(\varrho - 1)\right]. \tag{4.9}$$

Further assume that (3.1)– (3.5) holds and that the marginal law of Λ_ε corresponding to the second component converges to Λ weakly in the sense of measures on $L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)$. If $[(\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t), \mathbb{P}^\varepsilon); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon]$ is a finite energy weak martingale solution of (4.2) in the sense of Definition 3.2.6 with initial law Λ_ε, then

$$\begin{aligned} \varrho_\varepsilon &\to 1 \quad \textit{in law in} \quad L^\infty(0,T; L^{\overline{\gamma}}_{\mathrm{loc}}(\mathbb{R}^3)), \\ \mathbf{u}_\varepsilon &\to \mathbf{U} \quad \textit{in law in} \quad \big(L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)), w\big), \\ \varrho_\varepsilon\mathbf{u}_\varepsilon &\to \mathbf{U} \quad \textit{in law in} \quad L^2(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^3)) \end{aligned}$$

where $\overline{\gamma} = \min\{2, \gamma\}$, $q < 2\gamma/\gamma + 1$ and where $\mathbf{U}$ is a weak martingale solution of (4.3) in the sense of Definition 4.2.7 with the initial law Λ.

We now devote the entirety of the rest of this chapter to the proof of Theorem 4.2.9.

4.3 Uniform bounds

For every $\varepsilon > 0$, let assume there exits a finite energy weak martingale solution of Equation (4.2) given by

$$[(\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}_t^\varepsilon), \mathbb{P}^\varepsilon), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon] .$$

Indeed such an existence of a solution holds from the result shown in Chapter 3. Now in analogy to (3.41), the following energy inequality holds

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^3}\left(\frac{\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2}{2}+H(\varrho_\varepsilon)\right)(t)\,\mathrm{d}x\right]^p + \mathbb{E}\left[\int_{Q_T}\mathbb{S}(\nabla\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon\,\mathrm{d}x\mathrm{d}s\right]^p \\ \leq c_p\left(1+\mathbb{E}\left[\int_{\mathbb{R}^3}\left(\frac{|(\varrho_\varepsilon\mathbf{u}_\varepsilon)(0)|^2}{2\varrho_\varepsilon(0)}+H(\varrho_\varepsilon(0,\cdot))\right)\mathrm{d}x\right]^p\right), \end{aligned} \tag{4.10}$$

where $Q_T := (0,T)\times\mathbb{R}^3$ and now

$$\begin{aligned} H(\varrho_\varepsilon) := H(\varrho_\varepsilon, 1) &= P(\varrho_\varepsilon) - P'(1)(\varrho_\varepsilon - 1) - P(1) \\ &= \frac{1}{\varepsilon^2(\gamma-1)}\left[\varrho_\varepsilon^\gamma - 1 - \gamma(\varrho_\varepsilon - 1)\right] \end{aligned} \tag{4.11}$$

for

$$P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma, \quad a = \mathrm{Ma}^{-2} = \varepsilon^{-2}.$$

By using the initial law (4.8), we get

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{R}^3}\left(\frac{|(\varrho_\varepsilon\mathbf{u}_\varepsilon)(0)|^2}{2\varrho_\varepsilon(0)}+H(\varrho_\varepsilon(0))\right)\mathrm{d}x\right]^p \\ &= \int_{L_x^\gamma\times L_x^{\frac{2\gamma}{\gamma+1}}}\left\|\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+H(\varrho)\right\|_{L_x^1}^p \mathrm{d}\Lambda_\varepsilon(\varrho,\mathbf{m}) \leq c_p. \end{aligned} \tag{4.12}$$

Similar to (3.48)–(3.54), we can now collect the following global uniform (in ε) bounds

$$
\begin{aligned}
\mathbb{E}\left|\left(\int_0^T \left\|\nabla \mathbf{u}_\varepsilon\right\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}t\right)^{\frac{1}{2}}\right|^p &\leq c,\\
\mathbb{E}\left|\sup_{t\in[0,T]} \left\|H(\varrho_\varepsilon)\right\|_{L^1(\mathbb{R}^3)}\right|^p &\leq c,\\
\mathbb{E}\left|\sup_{t\in[0,T]} \left\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\right\|_{L^2(\mathbb{R}^3)}\right|^p &\leq c,
\end{aligned}
\tag{4.13}
$$

where $c = c(p)$, as well as the following local bounds

$$
\begin{aligned}
\mathbb{E}\left|\sup_{t\in[0,T]} \left\|\varrho_\varepsilon\mathbf{u}_\varepsilon\right\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p &\leq c,\\
\mathbb{E}\left|\left(\int_0^T \left\|\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\right\|^2_{L^{\frac{6\gamma}{4\gamma+3}}(B_r)} \mathrm{d}t\right)^{\frac{1}{2}}\right|^p &\leq c,
\end{aligned}
\tag{4.14}
$$

with $c = c(p,r)$ for any $p \in [1,\infty)$.

4.4 Strong convergence of density

The lemma below follows [59, Lemma 2.] and [42].

Lemma 4.4.1. Denote by $z \geq 0$, a density function. Then for all $\varepsilon \in (0,1)$, there exists constants $c_i > 0$, $i = 1,2,3$, independent of ε such that

$$|z-1|^2 \leq \varepsilon^2(\gamma-1)H(z), \quad \text{if } \gamma \geq 2 \text{ and } 0 \leq z < \infty, \tag{4.15}$$

$$c_1\,|z-1|^2 \leq \varepsilon^2 H(z) \leq c_2\,|z-1|^2, \quad \text{if } \gamma > 1 \text{ and } \frac{1}{2} < z < 2, \tag{4.16}$$

$$c_3 \ \leq \varepsilon^2 H(z), \quad \text{if } \gamma > 1 \text{ and } 0 \leq z \leq \frac{1}{2} \text{ or } z \geq 2 \tag{4.17}$$

where $H(z)$ is the isentropic pressure potential (4.11).

Proof. Before we start, let first recall from (4.11) that

$$\varepsilon^2(\gamma-1)H(z) = z^\gamma - 1 - \gamma(z-1). \tag{4.18}$$

Case 1a: Let $\gamma = 2$. Then we get that $\varepsilon^2 H(z) = (z-1)^2$.

Case 1b: Let $\gamma > 2$ and $z \in [0, \frac{1}{2}]$.

We consider $R(z) = z^\gamma - 1 - \gamma(z-1) - |z-1|^\gamma$ where $|z-1|^\gamma \in C^2\big([0,1) \cup (1,\infty)\big)$ is piecewise smooth.

Then $R(0) > 0$, $R(1/2) > 0$, $\gamma(\gamma-1) > 0$ and

$$R''(z) = \gamma(\gamma-1)z^{\gamma-2}\left[1 - \left|1 - \frac{1}{z}\right|^{\gamma-2}\right] < 0, \quad z \in \left(0, \frac{1}{2}\right). \tag{4.19}$$

since $|1 - \frac{1}{z}| > 1$. Thus $R(z)$ is strictly concave on $[0, \frac{1}{2}]$ and so non-negative. It follows from (4.18) that

$$\varepsilon^2(\gamma-1)H(z) > |z-1|^\gamma. \tag{4.20}$$

Case 1c: Let $\gamma > 2$ and $z \in (\frac{1}{2}, \infty)$.

Then $R(1) = R'(1) = 0$ and so $z = 1$ is a critical point. Furthermore, if $z \in (\frac{1}{2}, 1) \cup (1, \infty)$, then $R''(z) > 0$ since $0 < |1 - \frac{1}{z}| < 1$. Thus $R(z)$ is convex in $(\frac{1}{2}, \infty)$ and so

$$\varepsilon^2(\gamma-1)H(z) \geq |z-1|^\gamma. \tag{4.21}$$

Case 2a: Let $1 < \gamma < 2$ and $\frac{1}{2} < z < 2$. Then $f(y) = y^{\gamma-2}$ is a strictly decreasing function for any $y \in (\frac{1}{2}, 2)$.

On the other hand, Taylor's Theorem gives for some $y \in (1, z)$ or $y \in (z, 1)$,

$$z^\gamma - 1 - \gamma(z-1) = \gamma(\gamma-1)y^{\gamma-2}\frac{(z-1)^2}{2}. \tag{4.22}$$

It follows that

$$\gamma(\gamma-1)2^{\gamma-3}|z-1|^2 \leq \varepsilon^2(\gamma-1)H(z) \leq \gamma(\gamma-1)2^{1-\gamma}|z-1|^2. \tag{4.23}$$

Case 2b: Let $\gamma \geq 2$ and $\frac{1}{2} < z < 2$. Then $f(y) = y^{\gamma-2}$ is an increasing function for any $y \in (\frac{1}{2}, 2)$ and so

$$\gamma(\gamma-1)2^{1-\gamma}|z-1|^2 \leq \varepsilon^2(\gamma-1)H(z) \leq \gamma(\gamma-1)2^{\gamma-3}|z-1|^2. \tag{4.24}$$

Case 3: Let $1 < \gamma < 2$ and $z \in Z$ where $Z := \{0 \leq z \leq \frac{1}{2}\} \cup \{z \geq 2\}$.

A second derivative test shows that $z \mapsto z^\gamma - 1 - \gamma(z-1)$ is convex for any $z \geq 0$ with a minimum value zero occurring at $z = 1 \notin Z$.

We can therefore conclude that $G(z) = z^\gamma - 1 - \gamma(z-1)$ is strictly positive for any $z \in Z$ and thus

$$z^\gamma - 1 - \gamma(z-1) \geq G(\partial Z) \tag{4.25}$$

where $\partial Z = \frac{1}{2}$ when $0 \leq z \leq \frac{1}{2}$ or $\partial Z = 2$ when $z \geq 2$.

□

We are now in the position to show that the sequence of densities indeed converges to a constant in the limit. We state this is the lemma below

Lemma 4.4.2. Set $\varphi_\varepsilon := \frac{\varrho_\varepsilon - 1}{\varepsilon}$. For all $p \in [1, \infty)$ and any $K \Subset \mathbb{R}^3$,

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left\|\varphi_\varepsilon\right\|_{L^{\overline{\gamma}}(K)}\right]^p \leq c(p) \tag{4.26}$$

holds uniformly in ε with $\overline{\gamma} = \min\{2, \gamma\}$. Furthermore, the strong convergence

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left\|\varrho_\varepsilon - 1\right\|_{L^{\overline{\gamma}}(K)}\right]^p = 0 \tag{4.27}$$

holds as $\varepsilon \to 0$.

Proof. Fix $\delta > 0$ and let $K \Subset \mathbb{R}^3$ be arbitrary. Then for all $\varrho_\varepsilon \geq 0$, we can use

Lemma 4.4.1 to get for $1 < \gamma < 2$ that,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\int_K |\varrho_\varepsilon - 1|^\gamma \,\mathrm{d}x\right]^p &= \mathbb{E}\left[\sup_{t\in[0,T]}\int_K |\varrho_\varepsilon - 1|^\gamma \,\mathbb{1}_{\{|\varrho_\varepsilon - 1|\geq\delta\}}\,\mathrm{d}x\right]^p \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]}\int_K |\varrho_\varepsilon - 1|^\gamma \,\mathbb{1}_{\{|\varrho_\varepsilon - 1|<\delta\}}\,\mathrm{d}x\right]^p \\ &\leq c(\delta)\,\mathbb{E}\left[\sup_{t\in[0,T]}\int_K \left[\varrho_\varepsilon^\gamma - 1 - \gamma\,(\varrho_\varepsilon - 1)\right]\,\mathrm{d}x\right]^p + c(T, \mathrm{vol}(K))\,\delta^{\gamma p}.\end{aligned} \tag{4.28}$$

However since $\varrho_\varepsilon^\gamma - 1 - \gamma\,(\varrho_\varepsilon - 1) = \varepsilon^2(\gamma - 1)H(\varrho_\varepsilon)$, by using the uniform bound $(4.13)_2$, it follows that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\int_K \left[\varrho_\varepsilon^\gamma - 1 - \gamma\,(\varrho_\varepsilon - 1)\,\right]\mathrm{d}x\right]^p \\ \leq&\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^3} \left[\varrho_\varepsilon^\gamma - 1 - \gamma\,(\varrho_\varepsilon - 1)\right]\,\mathrm{d}x\right]^p \leq c(\gamma)\,\varepsilon^{2p}.\end{aligned} \tag{4.29}$$

On the other hand, if $\gamma \geq 2$ then we can use (4.15) to get

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^3} |\varrho_\varepsilon - 1|^2\,\mathrm{d}x\right]^p &\leq \varepsilon^{2p}(\gamma-1)^p\,\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^3} H(\varrho_\varepsilon)\,\mathrm{d}x\right]^p \\ &\leq c\,\varepsilon^{2p}.\end{aligned} \tag{4.30}$$

By combining (4.28), (4.29) and (4.30), we can pass to the limit $\varepsilon \to 0$ and then $\delta \to 0$ to get

$$\lim_{\varepsilon\to0}\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\varrho_\varepsilon - 1\right\|_{L^{\overline{\gamma}}(K)}\right]^p = 0 \tag{4.31}$$

since $K \Subset \mathbb{R}^3$ was chosen arbitrarily. The convergence (4.27) thus follow. □

4.5 Acoustic wave equation

Let $\Delta_{\mathbb{R}^3}^{-1}$ represent the inverse of the Laplace operator on $\mathbb{R}^3$ and let $\mathcal{Q} = \nabla\Delta_{\mathbb{R}^3}^{-1}\mathrm{div}$ and $\mathcal{P}$ be, respectively, the gradient and solenoidal parts according to Helmoltz decomposition. Then we observe that by setting $\varphi_\varepsilon = \frac{\varrho_\varepsilon - 1}{\varepsilon}$ and $\mathrm{Id} = \mathcal{Q} + \mathcal{P}$, we derive from equation (4.2) that for all $\phi \in C_c^\infty(\mathbb{R}^3)$ and $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$, the following equation

$$\big\langle \varepsilon\, \varphi_\varepsilon(t)\,,\, \phi \big\rangle = \big\langle \varepsilon\, \varphi_\varepsilon(0)\,,\, \phi \big\rangle + \int_0^t \big\langle \mathrm{div}\mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(t)\,,\, \phi \big\rangle \, \mathrm{d}s \tag{4.32}$$

and

$$\begin{aligned} \big\langle \varepsilon\, \mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(t)\,,\, \boldsymbol{\phi} \big\rangle &= \big\langle \varepsilon\, \mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(0)\,,\, \boldsymbol{\phi} \big\rangle - \int_0^t \big\langle \gamma \nabla \varphi_\varepsilon\,,\, \boldsymbol{\phi} \big\rangle \, \mathrm{d}s \\ &+ \int_0^t \big\langle \varepsilon\, \mathbf{F}_\varepsilon^{\mathcal{Q}}\,,\, \boldsymbol{\phi} \big\rangle \, \mathrm{d}s + \int_0^t \big\langle \varepsilon\, \mathcal{Q}\Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon) \mathrm{d}W\,,\, \boldsymbol{\phi} \big\rangle \end{aligned} \tag{4.33}$$

holds where

$$\mathbf{F}_\varepsilon^{\mathcal{Q}} = \nu \Delta \mathcal{Q} \mathbf{u}_\varepsilon + (\lambda + \nu) \nabla \mathrm{div}\, \mathbf{u}_\varepsilon - \mathrm{div}\mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^2} \nabla [\varrho_\varepsilon^\gamma - 1 - \gamma(\varrho_\varepsilon - 1)].$$

Notice that (4.32)–(4.33) are well-defined since from $(4.14)_2$ and the continuity of $\mathcal{Q}$, we have that

$$\mathrm{div}\mathcal{Q}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \in L^p(\Omega; L^2(0,T; W^{-1, \frac{6\gamma}{4\gamma+3}}(B_r))) \tag{4.34}$$

independently of ε but which may depend on the radius $r > 0$ of the ball. And that

$$\nu \Delta \mathcal{Q} \mathbf{u}_\varepsilon + (\lambda + \nu) \nabla \mathrm{div} \mathbf{u}_\varepsilon \in L^p(\Omega; L^2(0,T; W^{-1,2}(B_r))) \tag{4.35}$$

uniformly in ε by virtue of $(4.13)_1$. Lastly, the regularity of the pressure potential $(4.13)_2$ and that of the density fluctuation (4.26) means that for $\overline{\gamma} = \min\{2, \gamma\} > 1$, the bound

$$\tilde{\mathbb{E}} \left[\sup_{t \in [0,T]} \left\| \frac{1}{\varepsilon^2} \nabla [\varrho_\varepsilon^\gamma - 1 - \gamma(\varrho_\varepsilon - 1)] \right\|_{W^{-1,\overline{\gamma}}(B_r)} \right]^p \leq c(p, r) \tag{4.36}$$

hold uniformly in ε.

By combining (4.34), (4.35) and (4.36) with the following continuous embeddings

$$L^2\big(0,T;W^{-1,\frac{6\gamma}{4\gamma+3}}(B_r)\big) \hookrightarrow L^2\big(0,T;W^{-l,2}(B_r)\big),$$
$$L^2\big(0,T;W^{-1,2}(B_r)\big) \hookrightarrow L^2\big(0,T;W^{-l,2}(B_r)\big),$$
$$L^\infty\big(0,T;W^{-1,\overline{\gamma}}(B_r)\big) \hookrightarrow L^2\big(0,T;W^{-l,2}(B_r)\big)$$

which holds for $l > 5/2$ on the ball B_r of radius $r > 0$, we get that,

$$\mathbf{F}_\varepsilon^{\mathcal{Q}} \in L^p\left(\Omega; L^2(0,T;W^{-l,2}(B_r))\right) \tag{4.37}$$

uniformly in ε (may depend on r).

4.6 Compactness

To explore compactness for our uniformly bounded sequences, let first define the path space $\chi = \chi_\varrho \times \chi_{\mathbf{u}} \times \chi_{\varrho\mathbf{u}} \times \chi_W$ where

$$\chi_\varrho = C_w\left([0,T]; L^\gamma_{\mathrm{loc}}(\mathbb{R}^3)\right), \qquad \chi_{\mathbf{u}} = \left(L^2(0,T;W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)), w\right),$$
$$\chi_{\varrho\mathbf{u}} = C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)\right), \qquad \chi_W = C\left([0,T]; \mathfrak{U}_0\right),$$

and let

1. $\mu_{\varrho_\varepsilon}$ be the law of ϱ_ε on the space χ_ϱ,
2. $\mu_{\mathbf{u}_\varepsilon}$ be the law of $\mathbf{u}_\varepsilon$ on $\chi_{\mathbf{u}}$,
3. $\mu_{\mathcal{P}(\varrho_\varepsilon\mathbf{u}_\varepsilon)}$ be the law of $\mathcal{P}(\varrho_\varepsilon\mathbf{u}_\varepsilon)$ on the space $\chi_{\varrho\mathbf{u}}$,
4. μ_W be the law of W on the space χ_W,
5. μ^ε be the joint law of $\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathcal{P}(\varrho_\varepsilon\mathbf{u}_\varepsilon)$, and W on the space χ.

We can now show the following tightness results for the above families of laws.

Proposition 4.6.1. The family of measures $\{\mu_{\varrho_\varepsilon};\, \varepsilon \in (0,1)\}$, $\{\mu_{\mathbf{u}_\varepsilon};\, \varepsilon \in (0,1)\}$ and $\{\mu_W;\, \varepsilon \in (0,1)\}$ are tight on χ_ϱ, $\chi_{\mathbf{u}}$ and χ_W respectively.

Proof. This follows exactly as in Proposition 3.3.12, Proposition 3.3.11 and the fact that μ_W is tight since it is a Radon measure on the Polish space χ_W. □

Now note that unlike Proposition 4.6.1 above, the following result is not quite the same as Proposition 3.3.14 albeit arguably simpler. For completeness, we show the proof which follows [11, Proposition 3.6.].

Proposition 4.6.2. The family of measures $\{\mu_{\mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)};\, \varepsilon \in (0,1)\}$ is tight on $\chi_{\varrho\mathbf{u}}$.

Proof. For $t \in (0,T)$, we decompose $\mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(t)$ into two parts

$$\begin{aligned}
\mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(t) &= \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)(0) - \int_0^t \mathcal{P}\big[\mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu \Delta \mathbf{u}_\varepsilon\big]\,\mathrm{d}s \\
&\quad + \int_0^t \mathcal{P}\Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\,\mathrm{d}W(s) \\
&=: Y^\varepsilon(t) + Z^\varepsilon(t)
\end{aligned}$$

where $Z^\varepsilon(t)$ represents the stochastic forcing part and $Y^\varepsilon(t)$, the rest.

First of all, since $\mathcal{P}$ is a continuous operator in the class of Sobolev spaces, it follows from the arguments (3.101)–(3.103) leading the (3.104) that for $l > \frac{3}{2}$, we have that

$$\mathbb{E}\,\|Z^\varepsilon\|_{C^\vartheta\left([0,T];W^{-l,2}(\mathbb{R}^3)\right)} = \mathbb{E}\,\|Z^\varepsilon\|_{C^\vartheta\left([0,T];W^{-l,2}(K)\right)} \lesssim 1 \tag{4.38}$$

uniformly in ε for $\vartheta > 0$ small. In (4.39), $K \Subset \mathbb{R}^3$ is the support of the noise.

Now given that the regularities $(4.13)_1$ and $(4.14)_2$ holds, and that the embedding $L^1(B_r) \hookrightarrow W^{-l+1,2}(B_r)$ is continuous for $l > \frac{5}{2}$, we can use the continuity of $\mathcal{P}$ to obtain for $\theta \geq 1$,

$$\begin{aligned}
&\mathbb{E}\big\|Y^\varepsilon(t) - Y^\varepsilon(s)\big\|_{W^{-l,2}(B_r)}^\theta \\
&\quad = \mathbb{E}\left\|\int_s^t \mathcal{P}\big[\mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nu\Delta\mathbf{u}_\varepsilon\big]\,\mathrm{d}\sigma\right\|_{W^{-l,2}(B_r)}^\theta \\
&\quad \lesssim \mathbb{E}\left\|\int_s^t \mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)\,\mathrm{d}\sigma\right\|_{W^{-l,2}(B_r)}^\theta + \mathbb{E}\left\|\int_s^t \Delta\mathbf{u}_\varepsilon\,\mathrm{d}\sigma\right\|_{W^{-l,2}(B_r)}^\theta \\
&\quad \lesssim \mathbb{E}\left\|\int_s^t \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\,\mathrm{d}\sigma\right\|_{L^1(B_r)}^\theta + \mathbb{E}\left\|\int_s^t \nabla\mathbf{u}_\varepsilon\,\mathrm{d}\sigma\right\|_{L^1(B_r)}^\theta \\
&\quad \lesssim |t-s|^{\theta/2}
\end{aligned} \tag{4.39}$$

uniformly in ε. Notice that in the last step above, we have used the continuity of the embeddings $L^{\frac{6\gamma}{4\gamma+3}}(B_r) \hookrightarrow L^1(B_r)$ and $L^2(B_r) \hookrightarrow L^1(B_r)$ which holds since the ball B_r is bounded and $\gamma > \frac{3}{2}$.

By Kolmogorov continuity criterion, it follows from the estimate (4.39) that

$$\mathbb{E}\left\|Y^\varepsilon\right\|_{C^\vartheta\left([0,T];W^{-l,2}(\mathbb{R}^3)\right)} \lesssim 1 \tag{4.40}$$

uniformly in ε for $\vartheta \in (0, \frac{1}{2})$.

We can now combine (4.39) and (4.40) to gain

$$\mathbb{E}\left\|\mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)\right\|_{C^\vartheta\left([0,T];W^{-l,2}(\mathbb{R}^3)\right)} \lesssim 1 \tag{4.41}$$

uniformly in ε for $\vartheta \in (0, \frac{1}{2})$ and $l > \frac{5}{2}$. Finally, by making use of (4.41), $(4.14)_1$ and the compact embedding

$$L^\infty\big(0,T;L^{\frac{2\gamma}{\gamma+1}}(B_r)\big) \cap C^\vartheta\big([0,T];W^{-l,2}(B_r)\big) \hookrightarrow C_w\big([0,T];L^{\frac{2\gamma}{\gamma+1}}(B_r)\big),$$

see [87, Corollary B.2], which completes the proof. □

Having established Proposition 4.6.1 and Proposition 4.6.2, the following lemma follows just as was done in Proposition 3.3.16.

Lemma 4.6.3. The sets $\{\mu^\varepsilon; \varepsilon \in (0,1)\}$ is tight on χ.

Now similar to Proposition 3.3.17, we apply the Jakubowski–Skorokhod representation theorem, Theorem 2.4.29 to get the following proposition.

Proposition 4.6.4. There exists a subsequence $\{\mu^\varepsilon; \varepsilon \in (0,1)\}$ (not relabelled), a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ with χ-valued Borel measurable random variables $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon, \tilde{W}_\varepsilon)$, $\varepsilon \in (0,1)$, and $(\tilde{\varrho}, \tilde{\mathbf{U}}, \tilde{\mathbf{m}}, \tilde{W})$ such that

- the law of $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon, \tilde{W}_\varepsilon)$ is given by μ^ε, $\varepsilon \in (0,1)$,
- the law of $(\tilde{\varrho}, \tilde{\mathbf{U}}, \tilde{\mathbf{m}}, \tilde{W})$, denoted by μ is a Radon measure,
- $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon, \tilde{W}_\varepsilon)$ converges $\tilde{\mathbb{P}}$-a.s to $(\tilde{\varrho}, \tilde{\mathbf{U}}, \tilde{\mathbf{m}}, \tilde{W})$ in the topology of χ, i.e.

$$\begin{aligned} \tilde{\varrho}_\varepsilon \to \tilde{\varrho} \quad &\text{in} \quad C_w\left([0,T]; L^\gamma_{\mathrm{loc}}(\mathbb{R}^3)\right) && \tilde{\mathbb{P}}\text{-a.s.,} \\ \tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{U}} \quad &\text{in} \quad \left(L^2(0,T; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)), w\right) && \tilde{\mathbb{P}}\text{-a.s.,} \\ \tilde{\mathbf{m}}_\varepsilon \to \tilde{\mathbf{m}} \quad &\text{in} \quad C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)\right) && \tilde{\mathbb{P}}\text{-a.s.,} \\ \tilde{W}_\varepsilon \to \tilde{W} \quad &\text{in} \quad C\left([0,T]; \mathfrak{U}_0\right) && \tilde{\mathbb{P}}\text{-a.s.,} \end{aligned} \tag{4.42}$$

To extend this new probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ into a stochastic basis, we endow it with the family of $\tilde{\mathbb{P}}$-augmented canonical filtrations for the random variables defined on it. That is, we consider the collection

$$\sigma_t[\tilde{\varrho}_\varepsilon] = \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\varrho}_\varepsilon(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T],$$

$$\vdots$$

$$\sigma_t[\tilde{\beta}_k] = \bigcap_{s>t} \sigma\Big(\sigma\big(\tilde{\beta}_k(r);\, 0 \le r \le s\big) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\Big), \quad t \in [0,T],$$

and let

$$\begin{aligned} \tilde{\mathscr{F}}^\varepsilon_t &= \sigma\left(\sigma_t[\tilde{\varrho}_\varepsilon]\,,\, \sigma_t[\tilde{\mathbf{u}}_\varepsilon]\,,\, \bigcup_{k=1}^\infty \sigma_k\left[\tilde{\beta}^\varepsilon_k\right]\right), \quad t \in [0,T] \\ \tilde{\mathscr{F}}_t &= \sigma\left(\sigma_t[\tilde{\varrho}]\,,\, \sigma_t[\tilde{\mathbf{u}}]\,,\, \bigcup_{k=1}^\infty \sigma_k\left[\tilde{\beta}_k\right]\right), \quad t \in [0,T] \end{aligned} \tag{4.43}$$

be the filtration for $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon)$ and $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W})$ respectively.

4.7 Identification of the limit

We now verify that on this new probability space, our new pair of processes

$$[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t^\varepsilon), \tilde{\mathbb{P}}), \tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon] \text{ and } [(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \tilde{\mathbf{U}}, \tilde{W}]$$

are indeed finite energy weak martingale solutions and a weak martingale solution respectively of Equations (4.2) and (4.3).

Proposition 4.7.1. $[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t^\varepsilon)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon]$ is a finite energy weak martingale solution of Equation (4.2) with initial law Λ_ε for $\varepsilon \in (0,1)$.

Proof. The proof of this proposition is similar to Lemma 3.3.20. □

Consequently, as in Section 3.3.8, the uniform bounds shown in (4.13), (4.14), (4.26) and (4.37) earlier holds true for these corresponding random processes on this new space. In particular for a generic constant $c = c(p,r)$, we have that the following bounds

$$\begin{aligned}
\tilde{\mathbb{E}}\left| \sup_{t\in[0,T]} \left\|\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right\|_{L^{\frac{2\gamma}{\gamma+1}}(B_r)}\right|^p &\leq c, \\
\tilde{\mathbb{E}}\left| \sup_{t\in[0,T]} \left\|\tilde{\varphi}_\varepsilon\right\|_{L^{\min\{2,\gamma\}}(B_r)}\right|^p &\leq c, \\
\tilde{\mathbb{E}}\left| \left(\int_0^T \left\|\tilde{\mathbf{u}}_\varepsilon\right\|^2_{W^{1,2}(B_r)} \mathrm{d}t \right)^{\frac{1}{2}}\right|^p &\leq c, \\
\tilde{\mathbb{E}}\left| \left(\int_0^T \left\|\tilde{\mathbf{F}}^{\mathcal{Q}}_\varepsilon\right\|^2_{W^{-l,2}(B_r)} \mathrm{d}t \right)^{\frac{1}{2}}\right|^p &\leq c,
\end{aligned} \tag{4.44}$$

holds uniformly in ε for any $p \in [1,\infty)$ and ball B_r. Here $l > 5/2$, $\tilde{\varphi}_\varepsilon = \frac{\tilde{\varrho}_\varepsilon - 1}{\varepsilon}$ and

$$\begin{aligned}
\tilde{\mathbf{F}}^{\mathcal{Q}}_\varepsilon &= \nu\Delta\mathcal{Q}\tilde{\mathbf{u}}_\varepsilon + (\lambda+\nu)\nabla\mathrm{div}\,\tilde{\mathbf{u}}_\varepsilon - \mathrm{div}\mathcal{Q}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) \\
&\quad - \frac{1}{\varepsilon^2}\nabla[\tilde{\varrho}_\varepsilon^\gamma - 1 - \gamma(\tilde{\varrho}_\varepsilon - 1)].
\end{aligned} \tag{4.45}$$

We now verify that indeed the limit process satisfies Definition 4.2.7. This will complete the proof of Theorem 4.2.9.

Proposition 4.7.2. $[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\mathbf{U}}, \tilde{W}]$ is a weak martingale solution of Equation (4.3) with initial law Λ.

The proof of this proposition will follow from the following lemmas and propositions. The entirety of the rest of this chapter is thus devoted to the proof of this proposition.

Lemma 4.7.3. The quadruplet $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ is a stochastic basis with a complete right-continuous filtration. Furthermore, $\tilde{W}$ is an $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process.

Proof. The first part immediatialy follows from Proposition 4.6.4 and the construction of the filtrations (4.43). For the second part, we follow the ideas of [15, Page 115].

From Proposition 4.6.4, since the law of $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon, \tilde{W}_\varepsilon)$ is given by μ^ε for every $\varepsilon \in (0,1)$, it follows from Theorem 2.4.31 that for every $\varepsilon \in (0,1)$, the process

$$\tilde{W}_\varepsilon = \sum_{k\in\mathbb{N}} e_k \tilde{\beta}_k^\varepsilon$$

is a cylindrical Wiener process and the filtration

$$\tilde{\mathscr{F}}_t^\varepsilon = \sigma\left(\sigma_t[\tilde{\varrho}_\varepsilon]\,,\, \sigma_t[\tilde{\mathbf{u}}_\varepsilon]\,,\, \bigcup_{k=1}^{\infty} \sigma_k\left[\tilde{\beta}_k^\varepsilon\right]\right), \quad t \in [0,T]$$

is non-anticipative with respect to $\tilde{W}_\varepsilon$.

Additionally, since (4.42) holds, we can use Lemma 2.4.32 to pass to the limit $\varepsilon \to 0$ and gain that the filtration

$$\tilde{\mathscr{F}}_t = \sigma\left(\sigma_t[\tilde{\varrho}]\,,\, \sigma_t[\tilde{\mathbf{u}}]\,,\, \bigcup_{k=1}^{\infty} \sigma_k\left[\tilde{\beta}_k\right]\right), \quad t \in [0,T]$$

is non-anticipative with respect to $\tilde{W}$. Finally, since the canonical filtration is the minimal filtration on which the process $\tilde{W}$ is adapted, it follows from Corollary 2.4.34 that $\tilde{W}$ is an $(\tilde{\mathscr{F}}_t)$-cylindrical Wiener process. This completes the proof. □

Lemma 4.7.4. Let $B_r \subset \mathbb{R}^3$ be an arbitrary ball of radius $r > 0$. For every $q < 6$, the following $\tilde{\mathbb{P}}$-a.s. convergence holds:

$$\tilde{\varrho}_\varepsilon \to 1 \quad \text{in} \quad L^\infty(0,T;L^{\min\{2,\gamma\}}(B_r)), \tag{4.46}$$

$$\mathcal{P}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2(0,T;W^{-1,2}(B_r)), \tag{4.47}$$

$$\mathcal{P}\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2(0,T;L^q(B_r)). \tag{4.48}$$

Proof. The convergence result (4.46) is a direct consequence of (4.27) and Proposition 4.6.4.

For the proof of (4.47), we first note that since the joint laws of $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon))$ and $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon)$ coincide, we can conclude that $\tilde{\mathbb{P}}$-a.s., $\tilde{\mathbf{m}}_\varepsilon = \mathcal{P}(\varrho_\varepsilon \mathbf{u}_\varepsilon)$. Now by using the continuity property of the operator $\mathcal{P}$, $(4.44)_1$, (4.46) and Proposition 4.6.4, we can conclude that $\tilde{\mathbb{P}}$-a.s.

$$\mathcal{P}\left(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right) \rightharpoonup \tilde{\mathbf{U}} \quad \text{in} \quad L^2\left(0,T;L^{\frac{2\gamma}{\gamma+1}}(B_r)\right) \tag{4.49}$$

for any $r > 0$. Indeed after passing to the limit in the continuity equation $(4.2)_1$ using (4.46), we get that $\operatorname{div} \tilde{\mathbf{U}} = 0$ and subsequently, $\tilde{\mathbf{U}}$ is identified with $\tilde{\mathbf{m}}$. The claim then follows from uniqueness of limits together with the aforementioned properties. Also, the compact embedding $C_w\left([0,T];L^{\frac{2\gamma}{\gamma+1}}(B_r)\right) \hookrightarrow L^2\left(0,T;W^{-1,2}(B_r)\right)$ and (4.49) means that

$$\mathcal{P}\left(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right) \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2\left(0,T;W^{-1,2}(B_r)\right) \tag{4.50}$$

$\tilde{\mathbb{P}}$-a.s.. for any $r > 0$. This shows (4.47).

Lastly using the fact that $\operatorname{div} \tilde{\mathbf{U}} = 0$ and that $\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\mathbf{U}}$ in $L^2(0,T;W^{1,2}(B_r))$ $\tilde{\mathbb{P}}$-a.s., we can conclude that

$$\operatorname{div} \tilde{\mathbf{u}}_\varepsilon \rightharpoonup 0 \quad \text{in} \quad L^2\left(0,T;L^2(B_r)\right)$$

which suggests that the gradient part of velocity is $\tilde{\mathbb{P}}$-a.s. zero. It therefore follows that the solenoidal part

$$\mathcal{P}\left(\tilde{\mathbf{u}}_\varepsilon\right) \rightharpoonup \tilde{\mathbf{U}} \quad \text{in} \quad L^2\left(0,T;W^{1,2}(B_r)\right) \tag{4.51}$$

$\tilde{\mathbb{P}}$-a.s. Now combining the duality pair (4.50) and (4.51), we then get

$$\int_0^T \int_{B_r} \mathcal{P}\left(\tilde{\mathbf{u}}_\varepsilon\right) \cdot \mathcal{P}\left(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right) \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_{B_r} |\tilde{\mathbf{U}}|^2 \,\mathrm{d}x \,\mathrm{d}t \tag{4.52}$$

$\tilde{\mathbb{P}}$-a.s. and subsequently we can find a constant c that is independent of ε such that

$$\begin{aligned}
&\left| \int_0^T \int_{B_r} (|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon|^2 - \mathcal{P}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \cdot \mathcal{P}\tilde{\mathbf{u}}_\varepsilon)\mathrm{d}x \,\mathrm{d}t \right| \\
&\qquad \leq \int_0^T \int_{B_r} |\mathcal{P}\tilde{\mathbf{u}}_\varepsilon| \, |\mathcal{P}(\tilde{\mathbf{u}}_\varepsilon - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)| \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = \int_0^T \int_{B_r} |\mathcal{P}\tilde{\mathbf{u}}_\varepsilon|^2 \, |1 - \tilde{\varrho}_\varepsilon| \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad \leq c \int_0^T \|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon\|^2_{L^{\max\left\{2, \frac{2\gamma}{\gamma-1}\right\}}(B_r)} \|1 - \tilde{\varrho}_\varepsilon\|_{L^{\min\{2,\gamma\}}(B_r)} \,\mathrm{d}t \\
&\qquad \leq c \, \|\tilde{\mathbf{u}}_\varepsilon\|^2_{L^2(0,T;W^{1,2}(B_r))} \|1 - \tilde{\varrho}_\varepsilon\|_{L^\infty\left(0,T;L^{\min\{2,\gamma\}}(B_r)\right)} \\
&\qquad \to 0
\end{aligned} \tag{4.53}$$

since for $\max\left\{2, \frac{2\gamma}{\gamma-1}\right\} < \frac{2N}{N-2} = 6$, the embedding $W^{1,2}(B_r) \hookrightarrow L^{\frac{2N}{N-2}}(B_r)$ is continuous. Combining this with (4.46) gives the convergence to zero.

It follows from the above result (4.53) and (4.52) that

$$\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon\|_{L^2((0,T)\times B_r)} \to \|\tilde{\mathbf{U}}\|_{L^2((0,T)\times B_r)} \tag{4.54}$$

and hence

$$\mathcal{P}\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2((0,T) \times B_r). \tag{4.55}$$

$\tilde{\mathbb{P}}$-a.s. for any $r > 0$.

Now given (4.51) and the continuous embedding $L^2\left(0,T;W^{1,2}(B_r)\right) \hookrightarrow L^2\left(0,T;L^6(B_r)\right)$, we can interpolate between $L^2\left(0,T;L^6(B_r)\right)$ and the function space in (4.55) to get for $2 < q < 6$,

$$\begin{aligned}\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\|_{L^2\left(0,T;L^q(B_r)\right)} &\leq c\,\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\|^{\theta}_{L^2\left(0,T;L^6(B_r)\right)} \\ &\quad\times \|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\|^{1-\theta}_{L^2\left((0,T)\times B_r\right)} \\ &\leq c\,\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\|^{1-\theta}_{L^2\left((0,T)\times B_r\right)}\end{aligned} \tag{4.56}$$

where the constant c is uniform in ε and $\theta \in (0,1)$. Passing to the limit finally yields (4.48). $\square$

The lemma below is crucial to the proof of the strong convergence of the gradient part of momentum given in Proposition 4.7.6 below. The statement of the result and its proof follows exactly as in [42, Lemma 8.1] and [99, Lemma 2.2]. See also [36, Lemma 3.1]. We reproduce the proof stressing the dependence of the vector field on the additional random parameter.

Lemma 4.7.5. Let $B \subset \mathbb{R}^3$ be a bounded ball. Then there exists a constant $c = c(B)$ such that

$$\mathbb{E}\int_{\mathbb{R}} \|e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}]\|^2_{L^2(B)}\,\mathrm{d}t \;\leq c\,\mathbb{E}\,\|\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}$$

for any $\mathbf{v} \in L^2(\Omega\times\mathbb{R}^3)$.

Proof. Let δ be the Dirac distribution at zero and $\phi(x) \in C_c^\infty(\mathbb{R}^3)$ be such that $\phi|_B \equiv 1$. We also set in the following, $\mathbf{v} = \mathbf{v}(\omega,\cdot)$ for a.e. $\omega \in \Omega$ fixed . Then it follows that

$$\int_{\mathbb{R}} \|e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}]\|^2_{L^2(B)}\mathrm{d}t \leq \int_{\mathbb{R}} \|e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}\phi]\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}t. \tag{4.57}$$

Now let us recall the spacetime Plancherel's identity, see [42, Pg. 281],

$$\begin{aligned}\int_{\mathbb{R}} \|e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}\phi]\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}t &= \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left|e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}\phi]\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3} \widehat{\phi}(\xi-\eta)\delta(\tau-\sqrt{\gamma}|\eta|)\widehat{\mathbf{v}}(\eta)\,\mathrm{d}\eta\right|^2\mathrm{d}\xi\,\mathrm{d}\tau\end{aligned} \tag{4.58}$$

which follows from the transformation identity

$$\left| \int_{\mathbb{R}^3} \widehat{\phi}(\xi - \eta)\delta(\tau - \sqrt{\gamma}|\eta|)\widehat{\mathbf{v}}(\eta)\, \mathrm{d}\eta \right|^2 = \left| \int_{\{\tau = \sqrt{\gamma}|\eta|\}} \widehat{\phi}(\xi - \eta)\widehat{\mathbf{v}}(\eta)\, \mathrm{d}S_\eta \right|^2.$$

Then by Cauchy–Schwartz inequality,

$$\begin{aligned} \left| \int_{\{\tau = \sqrt{\gamma}|\eta|\}} \widehat{\phi}(\xi - \eta)\widehat{\mathbf{v}}(\eta)\, \mathrm{d}S_\eta \right|^2 &\leq \left(\int_{\{\tau = \sqrt{\gamma}|\eta|\}} |\widehat{\phi}(\xi - \eta)|\, \mathrm{d}S_\eta \right) \\ &\quad \times \left(\int_{\{\tau = \sqrt{\gamma}|\eta|\}} |\widehat{\phi}(\xi - \eta)||\widehat{\mathbf{v}}(\eta)|^2\, \mathrm{d}S_\eta \right) \\ &\leq c \int_{\{\tau = \sqrt{\gamma}|\eta|\}} |\widehat{\phi}(\xi - \eta)||\widehat{\mathbf{v}}(\eta)|^2\, \mathrm{d}S_\eta \end{aligned} \tag{4.59}$$

hold where the constant c which depends on the support of the test function follows from

$$\sup_{\tau \in \mathbb{R},\, \xi \in \mathbb{R}^3} \left\{ \int_{\{\tau = \sqrt{\gamma}|\eta|\}} |\widehat{\phi}(\xi - \eta)| \right\} \leq c.$$

By combining (4.58), (4.59) and Fubini's theorem, it follows that

$$\begin{aligned} \int_{\mathbb{R}} \| e^{i\sqrt{-\gamma\Delta}t}[\mathbf{v}\phi] \|^2_{L^2(\mathbb{R}^3)} \mathrm{d}t &\leq c \int_{\mathbb{R}^3} \int_{\mathbb{R}} \int_{\{\tau = \sqrt{\gamma}|\eta|\}} |\widehat{\phi}(\xi - \eta)||\widehat{\mathbf{v}}(\eta)|^2\, \mathrm{d}S_\eta\, \mathrm{d}\tau\, \mathrm{d}\xi \\ &\leq c \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\widehat{\phi}(\xi - \eta)||\widehat{\mathbf{v}}(\eta)|^2\, \mathrm{d}\eta\, \mathrm{d}\xi \\ &\leq c\, \|\mathbf{v}\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{4.60}$$

The result then follow from the monotonicity property of expectations and (4.58).

□

Finally, we show that the gradient part of momentum vanishes in the limit.

Proposition 4.7.6. The strong convergence below holds.

$$\mathcal{Q}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \to 0 \quad \text{in} \quad L^2(0, T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3))$$

$\tilde{\mathbb{P}}$-a.s..

Proof. To proof this proposition, we first recall that by Proposition 4.7.1, we can infer that the system

$$
\begin{aligned}
&\varepsilon\,\mathrm{d}\tilde{\varphi}_\varepsilon + \mathrm{div}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\mathrm{d}t = 0,\\
&\varepsilon\,\mathrm{d}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon) + \gamma\nabla\tilde{\varphi}_\varepsilon\mathrm{d}t = \varepsilon\,\tilde{\mathbf{F}}_\varepsilon\mathrm{d}t + \varepsilon\,\Phi(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\mathrm{d}\tilde{W}_\varepsilon,
\end{aligned}
\tag{4.61}
$$

holds in the distributional sense where

$$\tilde{\varphi}_\varepsilon = \big(\varphi_\varepsilon - 1\big)\varepsilon^{-1}, \tag{4.62}$$

$$\tilde{\mathbf{F}}_\varepsilon = \mathrm{div}\big[\mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon)\big] - \mathrm{div}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\otimes\tilde{\mathbf{u}}_\varepsilon) - \frac{1}{\varepsilon^2}\nabla[\tilde{\varrho}_\varepsilon^\gamma - 1 - \gamma(\tilde{\varrho}_\varepsilon - 1)]. \tag{4.63}$$

We now aim to derive a mild solution for a smoothened version of the system above with the quantities localized on balls. To do this, we consider the following family of smooth cut-off functions which is motivated by cut-offs introduced in [36].

$$
\begin{aligned}
&\eta_\varepsilon \in C_c^\infty(\mathbb{R}^3), \quad 0 \le \eta_\varepsilon \le 1, \quad \eta_\varepsilon \equiv 1 \text{ in } B_{\varepsilon^{-1}},\\
&\eta_\varepsilon = 0 \text{ if } |x| \ge 2\varepsilon^{-1}, \quad |\nabla\eta_\varepsilon| \le c\,\varepsilon \text{ for } c > 0.
\end{aligned}
\tag{4.64}
$$

We now mollify the product of this cut-off function and our functions in (4.61) by means of spatial convolution with the standard mollifier. That is, if v_ε is one of the functions in (4.61), we set

$$v_{\varepsilon,\kappa} = (\eta_\varepsilon v_\varepsilon) * \wp^\kappa \tag{4.65}$$

where $\wp^\kappa$ is the standard mollifier in space with radius κ.

A critical observation is that, (4.65) does not commute with the differential operators defined in space. As such, remainder terms appears when (4.61) is replaced by its mollified counterpart for functions of the form (4.65).

Let us now proceed with the details by first considering $(4.61)_1$. By multiply $(4.61)_1$ by the cut-off function and proceeding to mollify, it holds that

$$\varepsilon\,\mathrm{d}\,\tilde{\varphi}_{\varepsilon,\kappa} + \mathrm{div}(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_\kappa\,\mathrm{d}t = \tilde{\mathcal{R}}_{\varepsilon,\kappa} \tag{4.66}$$

where

$$\tilde{\mathcal{R}}_{\varepsilon,\kappa} = \big[\nabla\eta_\varepsilon \cdot (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa. \tag{4.67}$$

Also note that $\tilde{\varphi}_{\varepsilon,\kappa} = \big[\eta_\varepsilon \tilde{\varphi}_\varepsilon\big] * \wp^\kappa$ and $(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_\kappa = \big[\eta_\varepsilon(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa$. And since $\mathcal{Q} = \nabla\Delta_{\mathbb{R}^3}^{-1}\mathrm{div}$, if we define

$$\tilde{\Psi}_{\varepsilon,\kappa} := \Delta_{\mathbb{R}^3}^{-1}\mathrm{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_\kappa \tag{4.68}$$

so that $\nabla\tilde{\Psi}_{\varepsilon,\kappa} = \mathcal{Q}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_\kappa$, then we are able to rewrite (4.66) as

$$\varepsilon\,\mathrm{d}\,\tilde{\varphi}_{\varepsilon,\kappa} + \Delta\tilde{\Psi}_{\varepsilon,\kappa}\,\mathrm{d}t = \tilde{\mathcal{R}}_{\varepsilon,\kappa}. \tag{4.69}$$

In a similar manner, when we multiply $(4.61)_2$ by the cut-off function and mollify the system, we obtain

$$\varepsilon\,\mathrm{d}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_\kappa + \gamma\nabla\tilde{\varphi}_{\varepsilon,\kappa}\,\mathrm{d}t = \varepsilon\,\tilde{\mathbf{F}}_{\varepsilon,\kappa}\,\mathrm{d}t + \varepsilon\,\tilde{\mathbf{R}}_{\varepsilon,\kappa}\,\mathrm{d}t + \varepsilon\,\tilde{\Phi}_{\varepsilon,\kappa}\,\mathrm{d}\tilde{W}_\varepsilon \tag{4.70}$$

where

$$\tilde{\Phi}_{\varepsilon,\kappa} = \big[\eta_\varepsilon \Phi(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa, \tag{4.71}$$

$$\begin{aligned}\tilde{\mathbf{F}}_{\varepsilon,\kappa} &= \mathrm{div}\big[\eta_\varepsilon \mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa - \mathrm{div}\big[\eta_\varepsilon(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa \\ &\quad - \frac{1}{\varepsilon^2}\nabla\big[\eta_\varepsilon\big(\tilde{\varrho}_\varepsilon^\gamma - 1 - \gamma(\tilde{\varrho}_\varepsilon - 1)\big)\big] * \wp^\kappa\end{aligned} \tag{4.72}$$

and lower order remainder terms

$$\begin{aligned}\tilde{\mathbf{R}}_{\varepsilon,\kappa} = &-\big[\nabla\eta_\varepsilon \cdot \mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa + \big[\nabla\eta_\varepsilon \cdot (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon)\big] * \wp^\kappa + \frac{1}{\varepsilon}\big[\gamma\nabla\eta_\varepsilon\,\tilde{\varphi}_\varepsilon\big] * \wp^\kappa \\ &+ \frac{1}{\varepsilon^2}\big[\nabla\eta_\varepsilon\big(\tilde{\varrho}_\varepsilon^\gamma - 1 - \gamma(\tilde{\varrho}_\varepsilon - 1)\big)\big] * \wp^\kappa.\end{aligned} \tag{4.73}$$

We can now proceed to apply $\mathcal{Q}$ to (4.70) which yields

$$\varepsilon\,\mathrm{d}\nabla\tilde{\Psi}_{\varepsilon,\kappa} + \gamma\nabla\tilde{\varphi}_{\varepsilon,\kappa}\,\mathrm{d}t = \varepsilon\,\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\,\mathrm{d}t + \varepsilon\,\mathcal{Q}\tilde{\mathbf{R}}_{\varepsilon,\kappa}\,\mathrm{d}t + \varepsilon\,\mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa}\,\mathrm{d}\tilde{W}_\varepsilon. \tag{4.74}$$

We now rewrite (4.69) and (4.74) as the single system

$$\begin{aligned}\varepsilon \mathrm{d}\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}\end{bmatrix} &= \mathcal{A}\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}t + \varepsilon\begin{bmatrix}\frac{1}{\varepsilon}\tilde{\mathcal{R}}_{\varepsilon,\kappa}\\ \mathcal{Q}\tilde{\mathbf{R}}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}t\\ &+\varepsilon\begin{bmatrix}0\\ \mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}t + \varepsilon\begin{bmatrix}0\\ \mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}\tilde{W}_{\varepsilon}\end{aligned} \tag{4.75}$$

with the usual wave operator

$$\mathcal{A} = \begin{bmatrix}0 & -\mathrm{div}\\ -\gamma\nabla & 0\end{bmatrix}. \tag{4.76}$$

Now let $E = L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)$ and consider the operator given by $S(t) = e^{t\mathcal{A}}$. We observe that $(S(t))_{t\geq 0}$ is a strongly continuous semigroup since,

$$S(0) = \mathbb{1}, \quad S(t+s) = S(t)S(s), \quad \lim_{t\downarrow 0} S(t)\mathbf{f} = \mathbf{f}$$

for all $\mathbf{f} = [\varphi, \nabla\Psi]^T \in E$. Moreover $\mathcal{A}$ is linear and

$$\mathcal{A}\mathbf{f} = \lim_{t\downarrow 0}\frac{S(t)\mathbf{f}-\mathbf{f}}{t} = \left.\frac{\mathrm{d}S(t)\mathbf{f}}{\mathrm{d}t}\right|_{t=0} = e^{t\mathcal{A}}\mathcal{A}\mathbf{f}\,|_{t=0}$$

hence $\mathcal{A}$ is an infinitesimal generator of the strongly continuous semigroup $S(t)$ with domain

$$\begin{aligned}\mathrm{Dom}(\mathcal{A}) &= \left\{\mathbf{f}\in E \,:\, \lim_{t\downarrow 0}\frac{S(t)\mathbf{f}-\mathbf{f}}{t} \text{ exists }\right\}\\ &= \left\{\mathbf{f} = [\varphi,\nabla\Psi]^T \,:\, \varphi\in W^{1,2}(\mathbb{R}^3),\ \nabla\Psi\in L^2(\mathbb{R}^3),\ \mathrm{div}\,\nabla\Psi \in L^2(\mathbb{R}^3)\right\}.\end{aligned}$$

As such, the following proposition which is a special case of a standard theorem holds, cf. Theorem 4.2.5.

Proposition 4.7.7. Assume that $\mathcal{A} \,:\, \mathrm{Dom}(\mathcal{A}) \subset E \longrightarrow E$ is an infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on E and that

$$\tilde{\Phi}_{\varepsilon,\kappa} \in \mathcal{N}^2_W\big(0,T;L_2(\mathfrak{U};W^{-l,2}(\mathbb{R}^3))\big), \quad l>3/2,$$

recall (4.5) for the definition of this space. Then a weak solution of (4.75) is also a mild solution.

As a result of Proposition 4.7.7, we can rewrite Equation (4.75), after rescaling in time, in the mild form

$$\begin{aligned}\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}\end{bmatrix}(t) &= S\left(\frac{t}{\varepsilon}\right)\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix} + \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\begin{bmatrix}0\\ \mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}s\\ &\quad + \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\begin{bmatrix}\frac{1}{\varepsilon}\tilde{\mathcal{R}}_{\varepsilon,\kappa}\\ \mathcal{Q}\tilde{\mathbf{R}}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}s + \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\begin{bmatrix}0\\ \mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}\tilde{W}_{s,\varepsilon}\end{aligned} \tag{4.77}$$

where the semigroup $S(t)$ is such that

$$S\,(t)\begin{bmatrix}\tilde{\varphi}_0\\ \nabla\tilde{\Psi}_0\end{bmatrix} = \begin{bmatrix}\tilde{\varphi}\\ \nabla\tilde{\Psi}\end{bmatrix}(t) \tag{4.78}$$

is the solution to the homogeneous problem

$$\begin{aligned}\mathrm{d}(\tilde{\varphi}) + \Delta\tilde{\Psi}\,\mathrm{d}t &= 0,\\ \mathrm{d}\nabla\tilde{\Psi} + \gamma\nabla\tilde{\varphi}\,\mathrm{d}t &= 0,\\ \tilde{\varphi}(0) = \tilde{\varphi}_0; \quad \nabla\tilde{\Psi}(0) &= \nabla\tilde{\Psi}_0.\end{aligned} \tag{4.79}$$

Using Fourier transforms (in space), we obtain a solution of equation (4.79) which is given by the pair (see Example 2.1.2.)

$$\begin{aligned}\nabla\tilde{\Psi}(t,x) &= \frac{e^{i\sqrt{-\gamma\Delta}t}}{2}\left(\nabla\tilde{\Psi}_0(x) + \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\tilde{\varphi}_0(x)\right)\\ &\quad + \frac{e^{-i\sqrt{-\gamma\Delta}t}}{2}\left(\nabla\tilde{\Psi}_0(x) - \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\tilde{\varphi}_0(x)\right),\\ \tilde{\varphi}(t,x) &= \frac{e^{i\sqrt{-\gamma\Delta}t}}{2}\left(\tilde{\varphi}_0(x) - \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\tilde{\Psi}_0(x)\right)\\ &\quad + \frac{e^{-i\sqrt{-\gamma\Delta}t}}{2}\left(\tilde{\varphi}_0(x) + \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\tilde{\Psi}_0(x)\right).\end{aligned} \tag{4.80}$$

Remark 4.7.8. In the following, $D^{1,2}(\mathbb{R}^3) = \left(\sqrt{-\Delta}\right)^{-1}L^2(\mathbb{R}^3)$ corresponds to the homogeneous Sobolev space. See Keel and Tao [62, Page 957].

By using (4.78), (4.80) and Lemma 4.7.5, we obtain that

$$\tilde{\mathbb{E}}\left\|S(t)\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix}\right\|^2_{L^2(\mathbb{R}\times B)} \lesssim \tilde{\mathbb{E}}\left\|\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix}\right\|^2_{L^2(\mathbb{R}^3)} \tag{4.81}$$

uniformly in ε for any ball $B \subset \mathbb{R}^3$. So by rescaling in time, i.e, setting $s = \frac{t}{\varepsilon}$ so that $\mathrm{d}s = \frac{\mathrm{d}t}{\varepsilon}$, we get from (4.81) that

$$\begin{aligned}\tilde{\mathbb{E}}\left\|S\left(\frac{t}{\varepsilon}\right)\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix}\right\|^2_{L^2((0,T)\times B)} &\leq \tilde{\mathbb{E}}\left\|S\left(\frac{t}{\varepsilon}\right)\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix}\right\|^2_{L^2(\mathbb{R}\times B)}\\ &\lesssim \varepsilon\,\tilde{\mathbb{E}}\left\|\begin{bmatrix}\tilde{\varphi}_{\varepsilon,\kappa}(0)\\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(0)\end{bmatrix}\right\|^2_{L^2(\mathbb{R}^3)}\end{aligned} \tag{4.82}$$

with a constant that is independent of ε.

Remark 4.7.9. We emphasis that although the hidden constant in $\lesssim$ is independent of ε, it may, and would usually depend on the smoothing kernel κ and on the ball B. This remark applies to subsequent estimates below.

To continue, we make the following notations:

$$\tilde{\mathbf{m}}_\varepsilon := (\eta_\varepsilon \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \quad \text{and} \quad \tilde{\mathbf{m}}_{\varepsilon,\kappa} := (\eta_\varepsilon \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) * \wp^\kappa \tag{4.83}$$

and recall that $\nabla \tilde{\Psi}_{\varepsilon,\kappa} = \mathcal{Q}\tilde{\mathbf{m}}_{\varepsilon,\kappa}$. By the continuity of $\mathcal{Q}$, Young's inequality for convolution and the assumption on the initial law (4.7), we obtain

$$\tilde{\mathbb{E}} \big\| \nabla \tilde{\Psi}_{\varepsilon,\kappa}(0) \big\|^2_{L^2(\mathbb{R}^3)} \lesssim \tilde{\mathbb{E}} \big\| \tilde{\mathbf{m}}_\varepsilon(0) \big\|^2_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3)} \lesssim_\kappa 1 \tag{4.84}$$

and similarly,

$$\tilde{\mathbb{E}} \big\| \tilde{\varphi}_{\varepsilon,\kappa}(0) \big\|^2_{L^2(\mathbb{R}^3)} \lesssim_\kappa 1. \tag{4.85}$$

By substituting (4.84)–(4.85) in (4.82), we have shown that

$$\tilde{\mathbb{E}} \left\| S\left(\frac{t}{\varepsilon}\right) \begin{bmatrix} \tilde{\varphi}_{\varepsilon,\kappa}(0) \\ \nabla \tilde{\Psi}_{\varepsilon,\kappa}(0) \end{bmatrix} \right\|^2_{L^2((0,T)\times B)} \lesssim_\kappa \varepsilon. \tag{4.86}$$

Now we use Jensen's inequality and extend the time variable s from $(0,t)$ to $\mathbb{R}$ to get

$$\tilde{\mathbb{E}} \left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa} \mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \leq \tilde{\mathbb{E}} \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa} \right\|^2_{L^2(\mathbb{R}\times(0,T)\times B)}. \tag{4.87}$$

Here, the extension in time to $\mathbb{R}$ is because we wish to use Lemma 4.7.5, the proof of which is based on spacetime Fourier transform. The function is therefore to be understand as the extension by zero outside of the time interval on which it is defined.

Furthermore, we can use the semigroup property and apply a similar estimate as in

(4.82) to get

$$\begin{aligned}\tilde{\mathbb{E}}\left\|S\left(\frac{t-s}{\varepsilon}\right)\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\right\|^2_{L^2(\mathbb{R}\times(0,T)\times B)} &= \tilde{\mathbb{E}}\left\|S\left(\frac{t}{\varepsilon}\right)S\left(\frac{-s}{\varepsilon}\right)\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\right\|^2_{L^2(\mathbb{R}\times(0,T)\times B)} \\ &\lesssim \varepsilon\,\tilde{\mathbb{E}}\left\|S\left(\frac{-s}{\varepsilon}\right)\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\right\|^2_{L^2((0,T)\times\mathbb{R}^3)}\end{aligned} \tag{4.88}$$

Finally since $(S(t))_t$ is a group of isometries on L^2, see [42, Page 282], we can follow a similar argument as in (4.84) and obtain

$$\varepsilon\,\tilde{\mathbb{E}}\left\|S\left(\frac{-s}{\varepsilon}\right)\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\right\|^2_{L^2((0,T)\times\mathbb{R}^3)} = \varepsilon\,\tilde{\mathbb{E}}\left\|\mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\right\|^2_{L^2((0,T)\times\mathbb{R}^3)} \lesssim \varepsilon. \tag{4.89}$$

By (4.87)–(4.89), we have shown that

$$\tilde{\mathbb{E}}\left\|\int_0^t S\left(\frac{t-s}{\varepsilon}\right)\begin{bmatrix}0\\ \mathcal{Q}\tilde{\mathbf{F}}_{\varepsilon,\kappa}\end{bmatrix}\mathrm{d}s\right\|^2_{L^2((0,T)\times B)} \lesssim \varepsilon. \tag{4.90}$$

For the lower order remainder terms, we first notice that an analogous argument as (4.87)–(4.89) yields

$$\tilde{\mathbb{E}}\left\|\int_0^t S\left(\frac{t-s}{\varepsilon}\right)\frac{1}{\varepsilon}\tilde{\mathcal{R}}_{\varepsilon,\kappa}\mathrm{d}s\right\|^2_{L^2((0,T)\times B)} \lesssim \frac{1}{\varepsilon}\tilde{\mathbb{E}}\|\tilde{\mathcal{R}}_{\varepsilon,\kappa}\|^2_{L^2((0,T)\times B)}. \tag{4.91}$$

And by the definition of our cut-off function, in particular that $|\nabla\eta_\varepsilon| \lesssim \varepsilon$ as well as (4.67), it holds by using $(4.44)_1$ that

$$\frac{1}{\varepsilon}\tilde{\mathbb{E}}\|\tilde{\mathcal{R}}_{\varepsilon,\kappa}\|^2_{L^2((0,T)\times B)} \lesssim \varepsilon\,\tilde{\mathbb{E}}\|\tilde{\mathbf{m}}_{\varepsilon,\kappa}\|^2_{L^2((0,T)\times B)} \lesssim_\kappa \varepsilon. \tag{4.92}$$

Now, if we decompose (4.73) into $\tilde{\mathbf{R}}_{\varepsilon,\kappa} = \tilde{\mathbf{R}}^1_{\varepsilon,\kappa} + \tilde{\mathbf{R}}^2_{\varepsilon,\kappa}$ where

$$\tilde{\mathbf{R}}^2_{\varepsilon,\kappa} = \frac{\gamma}{\varepsilon}[\nabla\eta_\varepsilon\tilde{\varphi}_\varepsilon] * \wp^\kappa, \quad \tilde{\mathbf{R}}^1_{\varepsilon,\kappa} = \tilde{\mathbf{R}}_{\varepsilon,\kappa} - \tilde{\mathbf{R}}^2_{\varepsilon,\kappa},$$

that is,

$$\tilde{\mathbf{R}}^1_{\varepsilon,\kappa} = \left[\nabla\eta_\varepsilon \cdot \Big(-\mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon) + \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon + \frac{1}{\varepsilon^2}\big(\tilde{\varrho}^\gamma_\varepsilon - 1 - \gamma(\tilde{\varrho}_\varepsilon - 1)\big)\Big)\right] * \wp^\kappa, \tag{4.93}$$

then a comparison with (4.63) shows that

$$\tilde{\mathbf{R}}^1_{\varepsilon,\kappa} \approx \left[\nabla\eta_\varepsilon \cdot \nabla\Delta^{-1}_\varepsilon\,(\tilde{\mathbf{F}}_\varepsilon)\right] * \wp^\kappa$$

where the operator $\nabla\Delta^{-1}_\varepsilon \approx \mathrm{div}^{-1}$ is such that Δ^{-1}_ε is the fundamental solution of the Laplace equation defined on the support of the cut-off η_ε. By triangle inequality,

$$\tilde{\mathbb{E}}\left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\tilde{\mathbf{R}}_{\varepsilon,\kappa}\mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \lesssim I_1 + I_2 \tag{4.94}$$

where in analogy with (4.91)–(4.92) we can use (4.64) and Young's inequality for convolution to get that

$$I_1 := \tilde{\mathbb{E}}\left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\tilde{\mathbf{R}}^1_{\varepsilon,\kappa}\mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \lesssim \varepsilon^3\,\tilde{\mathbb{E}}\big\|\tilde{\mathbf{F}}_{\varepsilon,\kappa}\big\|^2_{L^2((0,T)\times B)} \lesssim_\kappa \varepsilon^3. \tag{4.95}$$

In the above, the boundedness of $\tilde{\mathbf{F}}_\varepsilon$ given by (4.63), follows from $(4.13)_{1,2}$ and $(4.14)_2$ since by Proposition 4.6.4, these estimates still holds on the new probability space.

Again similar to (4.91)–(4.92), one can use $(4.44)_2$ to show that

$$\begin{aligned} I_2 &= \tilde{\mathbb{E}}\left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\tilde{\mathbf{R}}^2_{\varepsilon,\kappa}\mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \\ &= \tilde{\mathbb{E}}\left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right)\frac{\gamma}{\varepsilon}[\nabla\eta_\varepsilon\tilde{\varphi}_\varepsilon] * \wp^\kappa\mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \\ &\lesssim \varepsilon\,\tilde{\mathbb{E}}\big\|\tilde{\varphi}_{\varepsilon,\kappa}\big\|^2_{L^2((0,T)\times B)} \lesssim \varepsilon. \end{aligned} \tag{4.96}$$

and from (4.92)–(4.96), we conclude that

$$\tilde{\mathbb{E}} \left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \begin{bmatrix} \frac{1}{\varepsilon}\tilde{\mathcal{R}}_{\varepsilon,\kappa} \\ \mathcal{Q}\tilde{\mathbf{R}}_{\varepsilon,\kappa} \end{bmatrix} \mathrm{d}s \right\|^2_{L^2((0,T)\times B)} \lesssim \varepsilon. \tag{4.97}$$

Now let make the notation $\tilde{\Phi}_{\varepsilon,\kappa}(e_i) := \left[\eta_\varepsilon \mathbf{g}_i(\cdot, \tilde{\varrho}_\varepsilon, \tilde{\mathbf{m}}_\varepsilon)\right]_\kappa =: \tilde{\mathbf{g}}_i^{\varepsilon,\kappa}$. We notice that the quantity $S(t)\mathcal{Q}\Phi$ is Hilbert–Schmidt if Φ is Hilbert–Schmidt by the continuity of $S(t)$ and $\mathcal{Q}$. As such, it follows from Fubini's theorem and Itô isometry that

$$\begin{aligned}
&\tilde{\mathbb{E}} \left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \begin{bmatrix} 0 \\ \mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa} \end{bmatrix} \mathrm{d}\tilde{W}_\varepsilon(s) \right\|^2_{L^2((0,T)\times B)} \\
&\quad = \tilde{\mathbb{E}} \int_0^T \int_0^t \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa} \right\|^2_{L_2(\mathfrak{U};B)} \mathrm{d}s\,\mathrm{d}t \\
&\quad = \tilde{\mathbb{E}} \int_0^T \int_0^t \sum_{i\in\mathbb{N}} \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(B)} \mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

And extending s from $(0,t)$ to $\mathbb{R}$ yields

$$\begin{aligned}
&\tilde{\mathbb{E}} \int_0^T \int_0^t \sum_{i\in\mathbb{N}} \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(B)} \mathrm{d}s\,\mathrm{d}t \\
&\qquad\qquad \leq \int_0^T \sum_{i\in\mathbb{N}} \int_{\mathbb{R}} \tilde{\mathbb{E}} \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(B)} \mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

where by the semigroup property and similar estimate as in (4.82),

$$\begin{aligned}
&\int_0^T \sum_{i\in\mathbb{N}} \int_{\mathbb{R}} \tilde{\mathbb{E}} \left\| S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(B)} \mathrm{d}s\,\mathrm{d}t \\
&\qquad = \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \left\| S\left(\frac{t}{\varepsilon}\right) S\left(\frac{-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(\mathbb{R}\times B)} \mathrm{d}t \\
&\qquad \lesssim \varepsilon \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \left\| S\left(\frac{-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|^2_{L^2(B)} \mathrm{d}t.
\end{aligned}$$

Since the semigroup is an isometry with respect to the L^2-norm and $\mathcal{Q}$ is a continuous operator, we get that

$$\begin{aligned}
\varepsilon \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \left\| S\left(\frac{-s}{\varepsilon}\right) \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|_{L^2(B)}^2 \mathrm{d}t &= \varepsilon \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \left\| \mathcal{Q}\tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|_{L^2(B)}^2 \mathrm{d}t \\
&\lesssim \varepsilon \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \left\| \tilde{\mathbf{g}}_i^{\varepsilon,\kappa} \right\|_{L^2(B)}^2 \mathrm{d}t \\
&\lesssim \varepsilon \, \tilde{\mathbb{E}} \int_0^T \sum_{i\in\mathbb{N}} \left\| \tilde{\mathbf{g}}_i^{\varepsilon} \right\|_{L^1(B)}^2 \mathrm{d}t \lesssim \varepsilon.
\end{aligned}$$

The last inequality follows from the assumption on the noise term (3.7) and the implicit bounds in (4.44).

We have therefore shown that

$$\tilde{\mathbb{E}} \left\| \int_0^t S\left(\frac{t-s}{\varepsilon}\right) \mathcal{Q}\tilde{\Phi}_{\varepsilon,\kappa} \mathrm{d}\tilde{W}_\varepsilon(s) \right\|_{L^2((0,T)\times B)}^2 \lesssim \varepsilon \tag{4.98}$$

where all constants hidden in $\lesssim$ are independent of ε but depends on κ and B. Combining (4.98) with the estimates (4.86), (4.90) and (4.97), we obtain from (4.77) that

$$\begin{aligned}
\tilde{\mathbb{E}} \left\| \begin{bmatrix} \tilde{\varphi}_{\varepsilon,\kappa}(t) \\ \nabla\tilde{\Psi}_{\varepsilon,\kappa}(t) \end{bmatrix} \right\|_{L^2((0,T)\times B)}^2 &= \tilde{\mathbb{E}} \left\| \tilde{\varphi}_{\varepsilon,\kappa}(t) \right\|_{L^2((0,T)\times B)}^2 \\
&\quad + \tilde{\mathbb{E}} \left\| \nabla\tilde{\Psi}_{\varepsilon,\kappa}(t) \right\|_{L^2((0,T)\times B)}^2 \\
&\lesssim \varepsilon.
\end{aligned} \tag{4.99}$$

So in particular,

$$\tilde{\mathbb{E}} \left\| \nabla\tilde{\Psi}_{\varepsilon,\kappa}(t) \right\|_{L^2((0,T)\times B)}^2 \lesssim \varepsilon. \tag{4.100}$$

Now we note that since $(4.44)_1$ holds uniformly in ε, for an arbitrary small $\delta > 0$, we can find a $\kappa(\delta)$ such that

$$\tilde{\mathbb{E}} \sup_{t\in[0,T]} \left\| \tilde{\mathbf{m}}_{\varepsilon,\kappa} - \tilde{\mathbf{m}}_\varepsilon \right\|_{L^{\frac{2\gamma}{\gamma+1}}(B)}^p \le \delta^p \le \delta \tag{4.101}$$

holds for any $1 \leq p < \infty$. We further deduce from (4.101) together with the continuous embedding $L^\infty(0,T;L^q(B)) \hookrightarrow L^2(0,T;L^q(B))$ where $q = \frac{2\gamma}{\gamma+1}$, and the continuity of $\mathcal{Q}$ that

$$\begin{aligned} \tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon,\kappa} - \nabla\tilde{\Psi}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} \leq \delta, \\ \tilde{\mathbb{E}}\big\|\tilde{\mathbf{m}}_{\varepsilon,\kappa} - \tilde{\mathbf{m}}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} \leq \delta \end{aligned} \tag{4.102}$$

for κ small enough and for an arbitrary choice of ε. It implies that

$$\lim_{\kappa\downarrow 0}\tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon,\kappa} - \nabla\tilde{\Psi}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} = 0, \quad q = \frac{2\gamma}{\gamma+1}.$$

Thus, combined with (4.100) and keeping in mind that $q < 2$, we have that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\downarrow 0}\tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} \\ &= \lim_{\varepsilon\downarrow 0}\lim_{\kappa\downarrow 0}\tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} \\ &\leq 2\lim_{\varepsilon\downarrow 0}\lim_{\kappa\downarrow 0}\tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon,\kappa} - \nabla\tilde{\Psi}_{\varepsilon}\big\|^2_{L^2(0,T;L^q(B))} \\ &+ 2\lim_{\varepsilon\downarrow 0}\lim_{\kappa\downarrow 0}\tilde{\mathbb{E}}\big\|\nabla\tilde{\Psi}_{\varepsilon,\kappa}\big\|^2_{L^2(0,T;L^q(B))} \\ &= 0 \end{aligned} \tag{4.103}$$

and since B is arbitrary, our claim follow.

□

Remark 4.7.10. We observe that by combining (4.47) and Proposition 4.7.6, we can only conclude that

$$\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2(0,T;W^{-1,2}(B)) \tag{4.104}$$

$\tilde{\mathbb{P}}$-a.s. on balls $B \subset \mathbb{R}^3$.

However, we can improve this spatial regularity. We give this as part of the lemma below.

Lemma 4.7.11. Let $\gamma > \frac{3}{2}$ and $l > \frac{3}{2}$. Then for all $q < 2\gamma/\gamma + 1$, we have that

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{U}} \quad \text{in} \quad L^2(0,T;L^q(B)), \tag{4.105}$$

$$\operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \operatorname{div}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}) \quad \text{in} \quad L^1(0,T;W^{-l,2}_{\operatorname{div}}(B)) \tag{4.106}$$

$\tilde{\mathbb{P}}$-a.s. for any ball $B \subset \mathbb{R}^3$.

Proof. For (4.105), by using the identity $\mathcal{P}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) = \mathcal{P}(\tilde{\varrho}_\varepsilon - 1)\tilde{\mathbf{u}}_\varepsilon + \mathcal{P}\tilde{\mathbf{u}}_\varepsilon$, the reverse triangle inequality and then the triangle inequality, we have that for any ball $B \subset \mathbb{R}^3$,

$$\begin{aligned}
\Big| \|\mathcal{P}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\|_{L^2(0,T;L^q(B))} &- \Big\|\tilde{\mathbf{U}}\Big\|_{L^2(0,T;L^q(B))} \Big| \\
&\leq \Big\|\mathcal{P}(\tilde{\varrho}_\varepsilon - 1)\tilde{\mathbf{u}}_\varepsilon + \mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\Big\|_{L^2(0,T;L^q(B))} \\
&\leq \|\mathcal{P}(\tilde{\varrho}_\varepsilon - 1)\tilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^q(B))} + \Big\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\Big\|_{L^2(0,T;L^q(B))} \\
&\leq c\Big\{ \|\tilde{\varrho}_\varepsilon - 1\|_{L^\infty(0,T;L^{\min\{2,\gamma\}}(B))} \|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(0,T;L^{\max\{2,\gamma'\}}(B))} \\
&+ \Big\|\mathcal{P}\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}}\Big\|_{L^2(0,T;L^q(B))} \Big\} \to 0
\end{aligned} \tag{4.107}$$

$\tilde{\mathbb{P}}$-a.s., where we have used $(4.44)_3$, (4.46), (4.48) and the continuity of $\mathcal{P}$. Also notice that $\gamma' = \frac{\gamma}{\gamma-1} < 6$ for $\gamma > \frac{6}{5}$ so that the continuous embedding

$$L^2(0,T;W^{1,2}(B)) \hookrightarrow L^2\Big(0,T;L^{\max\{2,\gamma'\}}(B)\Big)$$

holds true.

Combining (4.107) with Proposition 4.7.6 finishes the proof of (4.105).

For (4.106), given that $\tilde{\mathbf{U}}$ is solenoidal, it is enough to show that

$$\operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \mathcal{P}\tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \operatorname{div}(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}) \quad \text{in} \quad L^1(0,T;W^{-l,2}_{\operatorname{div}}(B)). \tag{4.108}$$

This follows directly from the combination of the strong convergence (4.48) and (4.105) and integration by parts.

□

With the collection of results above, the following lemma holds.

Lemma 4.7.12. For all $t \in [0, T]$ and $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$, we let

$$M(\varrho, \mathbf{u}, \mathbf{m})_t = \langle \mathbf{m}(t), \boldsymbol{\phi} \rangle - \langle \mathbf{m}(0), \boldsymbol{\phi} \rangle - \int_0^t \langle \mathbf{m} \otimes \mathbf{u}, \nabla \boldsymbol{\phi} \rangle \mathrm{d}s + \nu \int_0^t \langle \nabla \mathbf{u}, \nabla \boldsymbol{\phi} \rangle \mathrm{d}s \\ + (\lambda + \nu) \int_0^t \langle \operatorname{div} \mathbf{u}, \operatorname{div} \boldsymbol{\phi} \rangle \mathrm{d}s - \frac{1}{\varepsilon^2} \int_0^t \langle \varrho^\gamma, \operatorname{div} \boldsymbol{\phi} \rangle \mathrm{d}s.$$

Then $M(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t \to M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_t$ $\tilde{\mathbb{P}}$-a.s. as $\varepsilon \to 0$.

Remark 4.7.13. In particular Lemma 4.7.12 makes use of Proposition 4.6.4, Lemma 4.7.4, Lemma 4.7.5, Proposition 4.7.6, (4.46) and Lemma 4.7.11.

Proof of Proposition 4.7.2. The following lemma now completes the proof of Proposition 4.7.2.

Lemma 4.7.14. For all $t \in [0, T]$ and $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$, we define

$$N(\varrho, \mathbf{m})_t = \sum_{k \in \mathbb{N}} \int_0^t \langle \mathbf{g}_k(x, \varrho, \mathbf{m}) \, , \, \boldsymbol{\phi} \rangle^2 \mathrm{d}s, \quad N_k(\varrho, \mathbf{m})_t = \int_0^t \langle \mathbf{g}_k(x, \varrho, \mathbf{m}) \, , \, \boldsymbol{\phi} \rangle \mathrm{d}s.$$

Then we have that for $\varepsilon \in (0, 1)$

$$N(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t \to N(1, \tilde{\mathbf{U}})_t \quad \tilde{\mathbb{P}}\text{-a.s.},$$
$$N_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t \to N_k(1, \tilde{\mathbf{U}})_t \quad \tilde{\mathbb{P}}\text{-a.s.}$$

as $\varepsilon \to 0$.

Proof. By Minkowski's inequality, we have that

$$
\begin{aligned}
&\|\langle \Phi(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\cdot\,,\, \boldsymbol{\phi}\rangle - \langle \Phi(1, \tilde{\mathbf{U}})\cdot\,,\, \boldsymbol{\phi}\rangle\|_{L_2(\mathfrak{U};\mathbb{R})} \\
&= \left(\sum_{k\in\mathbb{N}} \left| \left\langle \left(\Phi(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) - \Phi(1, \tilde{\mathbf{U}}) \right)(e_k)\,,\, \boldsymbol{\phi} \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq c(\boldsymbol{\phi}) \left(\sum_{k\in\mathbb{N}} \int_{\mathrm{spt}(\boldsymbol{\phi})} \left| \mathbf{g}_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) - \mathbf{g}_k(1, \tilde{\mathbf{U}}) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq c \int_{\mathrm{spt}(\boldsymbol{\phi})} \left(\sum_{k\in\mathbb{N}} \left| \mathbf{g}_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) - \mathbf{g}_k(1, \tilde{\mathbf{U}}) \right|^2 \right)^{\frac{1}{2}} \mathrm{d}x.
\end{aligned}
$$

Now let $\mathbf{x} := (\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)$ and $\mathbf{y} := (1, \tilde{\mathbf{U}})$ be vectors in $\mathbb{R}^4$ and define the line segment joining them by

$$
L(\mathbf{x}, \mathbf{y}) = \{ t\mathbf{x} + (1+t)\mathbf{y} \,:\, 0 \leq t \leq 1 \}.
$$

Then by the mean value inequality, we can find $(\varrho, \mathbf{m}) \in L(\mathbf{x}, \mathbf{y})$ such that

$$
\begin{aligned}
&\int_{\mathrm{spt}(\phi)} \left(\sum_{k\in\mathbb{N}} \left| \mathbf{g}_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) - \mathbf{g}_k(1, \tilde{\mathbf{U}}) \right|^2 \right)^{\frac{1}{2}} \mathrm{d}x \\
&\leq \int_{\mathrm{spt}(\phi)} \left(\left| (\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) - (1, \tilde{\mathbf{U}}) \right|^2 \sum_{k\in\mathbb{N}} \left| \nabla_{\varrho,\mathbf{m}}\, \mathbf{g}_k(\varrho, \mathbf{m}) \right|^2 \right)^{\frac{1}{2}} \mathrm{d}x \\
&\leq c \left(\int_{\mathrm{spt}(\phi)} |\tilde{\varrho}_\varepsilon - 1|\, \mathrm{d}x + \int_{\mathrm{spt}(\phi)} \left| \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{U}} \right| \mathrm{d}x \right) \\
&=: I_1 + I_2
\end{aligned}
$$

where we have used (3.1)–(3.5).

Hence by using the embeddings $L^{\min\{2,\gamma\}} \hookrightarrow L^1$ and $L^q \hookrightarrow L^1$, which holds true for any compact set or ball in $\mathbb{R}^3$ and where q is as defined in Lemma 4.7.11, we get that $I_1 \to 0$ and $I_2 \to 0$ for a.e. (ω, t) in $\tilde{\Omega} \times (0, T)$. This is due to (4.46) and (4.105). Hence

$$
\langle \Phi(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)\cdot\,,\, \boldsymbol{\phi}\rangle \;\to\; \langle \Phi(1, \tilde{\mathbf{U}})\cdot\,,\, \boldsymbol{\phi}\rangle \quad \text{in} \quad L_2(\mathfrak{U};\mathbb{R}) \quad \tilde{\mathbb{P}} \times \mathcal{L}\text{-a.e.}
$$

which implies that

$$N(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t \;\to\; N(1, \tilde{\mathbf{U}})_t \quad \text{in} \quad L_2(\mathfrak{U}; \mathbb{R}) \quad \tilde{\mathbb{P}} \times \mathcal{L}\text{-a.e.}$$

Similar argument holds for $N_k(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_t \to N_k(1, \tilde{\mathbf{U}})_t \quad \tilde{\mathbb{P}}$-a.s.. □

Now for fixed times $s, t \in [0, T]$ such that $s < t$, we denote by $M(\cdot)_{s,t}$, the difference $M(\cdot)_t - M(\cdot)_s$ and similarly for $N(\cdot)_{s,t}$ and $N_k(\cdot)_{s,t}$. Let $\mathbf{r}_t$ be a continuous map that restrict functions to time t and h be a continuous function. Then by the equality of laws given by Proposition 4.6.4, we have that

$$\begin{aligned}
&\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon)\, [M(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t}] \\
&= \mathbb{E}\, h(\mathbf{r}_s\varrho_\varepsilon, \mathbf{r}_s\mathbf{u}_\varepsilon, \mathbf{r}_s W_\varepsilon)\, [M(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon)_{s,t}] = 0, \\
&\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon) \left[\left[M(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)^2\right]_{s,t} - N(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t} \right] \\
&= \mathbb{E}\, h(\mathbf{r}_s\varrho_\varepsilon, \mathbf{r}_s\mathbf{u}_\varepsilon, \mathbf{r}_s W_\varepsilon) \left[\left[M(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon)^2\right]_{s,t} - N(\varrho_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon)_{s,t} \right] = 0, \\
&\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\varrho}_\varepsilon, \mathbf{r}_s\tilde{\mathbf{u}}_\varepsilon, \mathbf{r}_s\tilde{W}_\varepsilon) \left[\left[M(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\tilde{\beta}_k^\varepsilon\right]_{s,t} - N(\tilde{\varrho}_\varepsilon, \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)_{s,t} \right] \\
&= \mathbb{E}\, h(\mathbf{r}_s\varrho_\varepsilon, \mathbf{r}_s\mathbf{u}_\varepsilon, \mathbf{r}_s W_\varepsilon) \left[\left[M(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon)\beta_k\right]_{s,t} - N(\varrho_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon)_{s,t} \right] = 0.
\end{aligned} \tag{4.109}$$

Using Lemma 4.7.12 and Lemma 4.7.14, we can now pass to the limit in equation (4.109) to gain

$$\begin{aligned}
\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\mathbf{U}}, \mathbf{r}_s\tilde{W}) \left[M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_{s,t} \right] &= 0, \\
\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\mathbf{U}}, \mathbf{r}_s\tilde{W}) \left[\left[M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})^2\right]_{s,t} - N(1, \tilde{\mathbf{U}})_{s,t} \right] &= 0, \\
\tilde{\mathbb{E}}\, h(\mathbf{r}_s\tilde{\mathbf{U}}, \mathbf{r}_s\tilde{W}) \left[\left[M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})\tilde{\beta}_k\right]_{s,t} - N_k(1, \tilde{\mathbf{U}})_{s,t} \right] &= 0.
\end{aligned} \tag{4.110}$$

Equation (4.110) means that $M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_t$ is an $(\tilde{\mathscr{F}}_t)$-martingale. Moreover, using $(4.110)_{2,3}$, we get the quadratic and cross-variation of $M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_t$ as

$$\begin{aligned}
\big\langle\big\langle M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_t \big\rangle\big\rangle &= N(1, \tilde{\mathbf{U}}), \\
\big\langle\big\langle M(1, \tilde{\mathbf{U}}, \tilde{\mathbf{U}})_t\,,\, \tilde{\beta}_k \big\rangle\big\rangle &= N_k(1, \tilde{\mathbf{U}})
\end{aligned}$$

which yields

$$\left\langle\left\langle M(1,\tilde{\mathbf{U}},\tilde{\mathbf{U}})_t - \int_0^t \langle \Phi(1,\tilde{\mathbf{U}})\,\mathrm{d}\tilde{W}\,,\,\boldsymbol{\phi}\rangle \right\rangle\right\rangle = 0.$$

That is, for $\boldsymbol{\phi} \in C^\infty_{c,\mathrm{div}}(\mathbb{R}^3)$ and $t \in [0,T]$, we have that

$$\begin{aligned}\langle \tilde{\mathbf{U}}(t)\,,\,\boldsymbol{\phi}\rangle = \langle \tilde{\mathbf{U}}(0)\,,\,\boldsymbol{\phi}\rangle &+ \int_0^t \langle \tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}\,,\,\nabla\boldsymbol{\phi}\rangle \mathrm{d}s - \nu \int_0^t \langle \nabla\tilde{\mathbf{U}}\,,\,\nabla\boldsymbol{\phi}\rangle \mathrm{d}s \\ &+ \int_0^t \langle \Phi(1,\tilde{\mathbf{U}})\,\mathrm{d}\tilde{W}\,,\,\boldsymbol{\phi}\rangle\end{aligned}$$

$\tilde{\mathbb{P}}$-a.s.

$\square$

Chapter 5

The inviscid limit result

5.1 Introduction

In this chapter, we are concerned with compressible fluids defined on the whole space $\mathbb{R}^3$. We still consider the time interval $[0,T]$ with $T > 0$ and $(\Omega, \mathscr{F}, \mathbb{P})$ remains our probability space. However, the same analysis applies on $\mathbb{R}^n$ with $n = 1, 2, 3$ provided the assumption on the adiabatic exponent in the pressure is replaced with the condition $\gamma > \frac{n}{2}$.

A fluid flow with negligible or no viscosity is called an *inviscid fluid*. For such fluids, the viscous stress tensor (1.10) in the Navier–Stokes system (1.16) is omitted. So when stochastic forces are taken into account, and for simplicity, the conservative deterministic force $\varrho\mathbf{f}$ appearing in (1.8) is ignore or assumed to be incorporated in the stochastic forcing term (with $\omega \in \Omega$ fixed, say), then we obtain the *stochastic Euler system*:

$$\begin{aligned} \mathrm{d}\varrho + \mathrm{div}(\varrho\mathbf{u})\mathrm{d}t &= 0, \\ \mathrm{d}(\varrho\mathbf{u}) + \big[\mathrm{div}\big(\varrho\mathbf{u}\otimes\mathbf{u}\big) + \nabla p(\varrho)\big]\,\mathrm{d}t &= \Phi(\varrho, \varrho\mathbf{u})\,\mathrm{d}W. \end{aligned} \tag{5.1}$$

Here, $p(\varrho) = a\varrho^\gamma$ with $a > 0$ remains the isentropic pressure just as in (1.16), and as already mentioned, with the choice of adiabatic exponent $\gamma > \frac{3}{2}$. This choice is for technical reason but in general, the simpler condition $\gamma > 1$ suffices in any dimension.

Also, when we endow the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$, then we assume that W is an $(\mathscr{F}_t)$-cylindrical Wiener process just as for the Navier–Stokes system. In fact, for the purpose of this chapter, the choice of W has to coincide for both the Euler and Navier–Stokes system. The assumptions on the driving process and the diffusion coefficient Φ will be made precise in Section 5.2.2.

We complement (5.1) with the far field condition

$$\varrho(x) \to \overline{\varrho}, \quad \mathbf{u} \to 0, \quad |x| \to \infty \tag{5.2}$$

for some $\overline{\varrho} > 0$. The initial conditions are random variables

$$\varrho(0,\cdot) = \varrho_0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

with sufficient spatial regularity specified later.

We aim in this chapter, to show the relationship between a variant of the stochastically forced compressible Navier–Stokes system (1.16) and the stochastically forced Euler system (5.1) where both are defined on the whole space $\mathbb{R}^3$.

To do this, we require a rescaled version of the stochastic compressible Navier–Stokes (1.16). If $(\rho, \mathbf{v})$ are the density and velocity satisfying (1.16) in $\mathbb{R}^3$, then similar to Section 4.1, we are interested in the spacetime transformation that leads to the following mappings

$$\rho \mapsto \varrho_\varepsilon, \quad \mathbf{v} \mapsto \mathbf{u}_\varepsilon, \quad \nu \mapsto \varepsilon\nu, \quad \lambda \mapsto \varepsilon\lambda \tag{5.3}$$

where the parameter $\varepsilon \in (0,1]$ corresponds to the inverse of the *Reynolds number* Re. The Reynolds number Re is a dimensionless quantity that essentially measures the level of viscosity in a fluid. This relationship between Re and the fluid viscosity is reciprocal so that when Re is large, viscosity is small and subsequently, the fluid is modelled by the Euler system (5.1).

The above transformation (5.3) leads to the scaled Navier–Stokes system

$$\begin{aligned} \mathrm{d}\varrho_\varepsilon + \mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon)\mathrm{d}t = 0, \\ \mathrm{d}(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \Big[\mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p(\varrho_\varepsilon)\Big]\mathrm{d}t = \varepsilon\, \mathrm{div}\, \mathbb{S}(\nabla \mathbf{u}_\varepsilon)\, \mathrm{d}t + \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\mathrm{d}W. \end{aligned} \tag{5.4}$$

Just as in (5.1), $p(\varrho_\varepsilon) = a\varrho_\varepsilon^\gamma$, $a > 0$ is the pressure for any such $\varepsilon \in (0, 1]$ with adiabatic exponent $\gamma > \frac{3}{2}$ and the viscous stress tensor is

$$\mathbb{S}(\nabla \mathbf{u}_\varepsilon) := \nu \left(\nabla \mathbf{u}_\varepsilon + \nabla^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}\, \mathbf{u}_\varepsilon \mathbb{I}\right) + \lambda\, \mathrm{div}\, \mathbf{u}_\varepsilon \mathbb{I} \tag{5.5}$$

with viscosity coefficients $\nu > 0$, $\lambda \geq 0$.

The relationship between the Navier–Stokes system and the Euler system that we wish to show is a comparison between a class of *weak solutions* of (5.4) and a *strong solution* to (5.1). The precise formulations will be given in Definition 5.2.4 and Definition 5.2.8 respectively but we first give a brief description here.

The class of weak solutions to the singular system (5.4) would be in analogy to Definition 3.2.5 for the non-singular Navier–Stoke system. Recall that such solutions are weak in the probabilistic sense (the probability space is an integral part of the solution) and also weak in the analytical sense (derivatives only exists in the sense of distributions). So note that in this particular case, the distributional form (3.11) would have a singular parameter in the viscosity components. Notice that by Sobolev's theorems, the functions in (5.4) may have stronger regularities (depending on the spatial dimension, say) than may appear at first glance when we work in $\mathbb{R}^n$.

On the other hand, we will consider a unique strong solution of (5.1) that is strong in both the analytical and probabilistic sense but only exists up to a stopping time. Here, a strong solution in the analytical sense refers to (5.1) being satisfied pointwise without having to test against test functions. And a strong solution in the probabilistic sense refers to (5.1) being defined on a fixed probability space.

The existence and uniqueness of this local strong solution to (5.1) was recently shown in [17] and we give the precise statement in Theorem 5.2.9 below. Corresponding results in the deterministic case are classical and we refer to [1] and [6].

As in the incompressible case, global existence and uniqueness for the Euler system is a famous open problem. The presence of noise does not seem to change the situation. As solutions to nonlinear hyperbolic systems are known to develop singularities in finite time, the question about global well-posedness in the class of weak solutions has been analysed extensively. This is based on the method of convex integration which has been developed in the context of fluid mechanics by De Lellis and Székelyhidi [25]. The non-uniqueness of global-in-time weak solutions to (5.1) has recently been shown in [13] proceeding similar result in the deterministic case, cf. [35].

Our main result in this chapter, stated in Theorem 5.2.11, shows that any sequence of weak solutions to (5.4) having finite energy converges locally in time to the unique strong solution of (5.1) as the viscosity coefficients ν and λ becomes small. A similar strategy has been employed in [12] in order to study the inviscid-incompressible limit (where in addition, $a = \frac{1}{\varepsilon^2}$ with $\varepsilon \to 0$ is considered, where the limit system is the incompressible Euler system). A major difference to [12] is the generality of the noise coefficients we consider here. Due to the incompressibility constraint on the target system only linear noise can be considered in [12]. In contrast, in the compressible case, we can allow the full generality for the noise for which the existence theory applies.

The main tool in our proof is the relative energy inequality introduced in Section 3.6 and whose origin has been discussed in Section 1.2.

5.2 Preliminaries

We now collect some assumptions and definitions pertinent to the analysis in this chapter.

5.2.1 Initial data

We consider the following ill-prepared data that connects the inputs of the Navier–Stokes and Euler systems. We assume that the initial data $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ of the system (5.4) is a $\mathscr{F}_0$-measurable pair of random variables which satisfies the following conditions

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^3} H\big(\varrho_{0,\varepsilon}, \overline{\varrho}\big) \,\mathrm{d}x < \infty, \quad \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 \in L^1(\mathbb{R}^3), \\ &0 < \varrho^- \leq \varrho_{0,\varepsilon} \leq \varrho^+ \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{5.6}$$

where ϱ^- and ϱ^+ are independent of ε and the function $H(\cdot,\cdot)$, given by

$$H(\varrho, \overline{\varrho}) = \frac{a}{\gamma - 1} \big[\varrho^\gamma - \gamma \overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho}) - \overline{\varrho}^\gamma\big], \tag{5.7}$$

is the pressure potential.

The initial data $(\varrho_0, \mathbf{u}_0)$ of the expected limit system (5.1) satisfy

$$(\varrho_0, \mathbf{u}_0) \in \overline{\varrho} + W^{s,2}(\mathbb{R}^3) \times W^{s,2}(\mathbb{R}^3), \quad \varrho_0 \geq \varrho^- > 0 \quad \mathbb{P}\text{-a.s.} \tag{5.8}$$

for $s \in \mathbb{R}$ to be specified later in the chapter.

Finally, we suppose that

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^3} H\big(\varrho_{0,\varepsilon}, \varrho_0\big) \,\mathrm{d}x \xrightarrow{\varepsilon \searrow 0} 0, \\ &\mathbb{E} \int_{\mathbb{R}^3} |\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2 \,\mathrm{d}x \xrightarrow{\varepsilon \searrow 0} 0. \end{aligned} \tag{5.9}$$

5.2.2 Stochastic framework

The driving process W in (5.1) and (5.4) is a unique cylindrical $(\mathscr{F}_t)$-Wiener process on a separable Hilbert space $\mathfrak{U}$ defined on the stochastic basis $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\big)$ with a complete, right-continuous filtration.

To give the precise definition of the diffusion coefficient Φ, consider $\varrho \in L^2(\mathbb{R}^3)$,

$\varrho \geq 0$, $\mathbf{m} \in L^2(\mathbb{R}^3)$ and define it as follows

$$\Phi(\varrho, \mathbf{m})e_k = \mathbf{g}_k(\cdot, \varrho(\cdot), \mathbf{m}(\cdot)).$$

We suppose that the coefficients $\mathbf{g}_k : \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ are C^s-functions that satisfy uniformly in $x \in \mathbb{R}^3$

$$\mathbf{g}_k(\cdot, 0, 0) = 0, \tag{5.10}$$

$$|\nabla^l \mathbf{g}_k(\cdot, \cdot, \cdot)| \leq \alpha_k, \quad \sum_{k \in \mathbb{N}} \alpha_k < \infty \quad \text{for all } l \in \{1, ..., s\}, \tag{5.11}$$

for some $s \in \mathbb{N}$. Finally, we assume that the $\mathbf{g}_k$s are compactly supported in space, i.e. there is $K \Subset \mathbb{R}^3$ such that

$$\operatorname{spt}(\mathbf{g}_k) \Subset K \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.12}$$

This is also assumed in the Navier–Stokes case in view of the far field condition (5.2), c.f. Chapter 3. A typical example we have in mind is

$$\mathbf{g}_k(x, \varrho, \mathbf{m}) = \mathbf{a}_k(x)\varrho + \mathbb{A}_k(x)\mathbf{m}, \tag{5.13}$$

where $\mathbf{a}_k : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbb{A}_k : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ are smooth functions which are compactly supported. However, our analysis applies to general nonlinear coefficients $\mathbf{g}_k$.

5.2.3 Concepts of solution

Here, we state the notion of a solution for the Navier–Stokes system (5.4) on $\mathbb{R}^3$.

Definition 5.2.4. Let $\overline{\varrho} > 0$ and $\varepsilon > 0$. Let Λ_ε be a family of Borel probability measures on $L^\gamma_{\text{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\text{loc}}(\mathbb{R}^3)$. Then $\left[\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\right); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W\right]_{\varepsilon>0}$ is a family of *finite energy weak martingale solutions* of (5.4) if

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;

2. W is a $(\mathscr{F}_t)$-cylindrical Wiener process;

3. the densities ϱ_ε satisfies $\varrho_\varepsilon \geq 0$, $t \mapsto \langle \varrho_\varepsilon(t,\cdot), \phi \rangle \in C([0,T])$ for any $\phi \in C_c^\infty(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho_\varepsilon(t,\cdot), \phi \rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot)\|^p_{L^\gamma(K)} \right] < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

4. the velocity fields $\mathbf{u}_\varepsilon$ are $(\mathscr{F}_t)$-adapted random distributions and

$$\mathbb{E}\left[\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(K)} \,\mathrm{d}t \right]^p < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

5. the momenta $\varrho_\varepsilon \mathbf{u}_\varepsilon$ satisfies $t \mapsto \langle \varrho_\varepsilon \mathbf{u}_\varepsilon, \boldsymbol{\varphi} \rangle \in C([0,T])$ for any $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho_\varepsilon \mathbf{u}_\varepsilon(t,\cdot), \boldsymbol{\phi} \rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|^p_{L^{\frac{2\gamma}{\gamma+1}}(K)} \right] < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

6. there exists $\mathscr{F}_0$-measurable random variables $\big(\varrho_{\varepsilon,0}\,,\ \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\big) = \big(\varrho_\varepsilon(0), \varrho_\varepsilon \mathbf{u}_\varepsilon(0)\big)$ such that $\Lambda_\varepsilon = \mathbb{P} \circ \big(\varrho_{\varepsilon,0}, \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\big)^{-1}$,

7. for all $\psi \in C_c^\infty(\mathbb{R}^3)$ and $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$ and all $t \in [0,T]$, the following

$$\begin{aligned}
\big\langle \varrho_\varepsilon(t), \psi \big\rangle &= \big\langle \varrho_{\varepsilon,0}, \psi \big\rangle - \int_0^t \big\langle \varrho_\varepsilon \mathbf{u}_\varepsilon, \nabla\psi \big\rangle \,\mathrm{d}s \\
\big\langle (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t), \boldsymbol{\phi} \big\rangle &= \big\langle \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}, \boldsymbol{\phi} \big\rangle - \int_0^t \big\langle \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \nabla\boldsymbol{\phi} \big\rangle \,\mathrm{d}s \\
&\quad + \varepsilon \int_0^t \big\langle \mathbb{S}(\nabla \mathbf{u}_\varepsilon)\,, \nabla\boldsymbol{\phi} \big\rangle \mathrm{d}s - \int_0^t \big\langle \varrho_\varepsilon^\gamma, \operatorname{div}\boldsymbol{\phi} \big\rangle \,\mathrm{d}s \\
&\quad + \int_0^t \big\langle \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon), \boldsymbol{\phi} \big\rangle \,\mathrm{d}W
\end{aligned} \tag{5.14}$$

hold $\mathbb{P}$-a.s.;

8. the energy inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H(\varrho_\varepsilon, \overline{\varrho})\right] \mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \le \int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon}, \overline{\varrho})\right] \mathrm{d}x \\
&\qquad + \frac{1}{2}\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{R}^3} \varrho_\varepsilon^{-1} |\mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{R}^3} \mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\beta_k
\end{aligned} \tag{5.15}$$

holds for a.e. $t \in [0,T]$ $\mathbb{P}$-a.s.

Theorem 5.2.5. *Let $\varepsilon > 0$ be fixed and let $\overline{\varrho} > 0$, $\gamma > \frac{3}{2}$. Assume that Λ_ε is a Borel probability measure on $L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3)$ such that*

$$\Lambda_\varepsilon \left\{ (\varrho, \mathbf{m}) \in L^\gamma_{\mathrm{loc}}(\mathbb{R}^3) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathbb{R}^3) \;:\quad 0 < M_1 \le \varrho \le M_2 \;\; a.e \right\} = 1$$

for constants $M_1, M_2 > 0$. Also assume that the following moment estimate

$$\int_{L^\gamma_x \times L^{\frac{2\gamma}{\gamma+1}}_x} \left\| \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + H(\varrho, \overline{\varrho}) \right\|^p_{L^1_x} \mathrm{d}\Lambda_\varepsilon(\varrho, \mathbf{m}) < \infty$$

holds for all $1 \le p < \infty$. Furthermore, assume that (5.10)–(5.12) *holds with $s = 1$. Then there exists a finite energy weak martingale solution of* (5.4) *in the sense of Definition 5.2.4 with initial law Λ_ε.*

Remark 5.2.6. Although Theorem 5.2.5 was shown in Chapter 3 specifically, Theorem 3.2.12 for $n = 3$, it also applies in general $n \le 3$ dimensions by replacing the bound $\gamma > \frac{3}{2}$ by $\gamma > \frac{n}{2}$.

As already mentioned, we aim to study a singular limit result satisfying (5.1) after passing to the limit.

The following pair of definition combines to give the notion of a solution to (5.1) that we have in mind.

Definition 5.2.7. Let $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \mathbb{P}\right)$ be a stochastic basis with a complete right-continuous filtration. Let W be an $(\mathscr{F}_t)$-cylindrical Wiener process and let

$\overline{\varrho} > 0$. Let $(\varrho_0, \mathbf{u}_0)$ be a $\overline{\varrho} + W^{s,2}(\mathbb{R}^3) \times W^{s,2}(\mathbb{R}^3)$-valued $(\mathscr{F}_0)$-measurable random variable and let Φ satisfy (5.10)–(5.12) for some $s \in \mathbb{N}$. We say that $(\varrho, \mathbf{u}, \mathtt{t})$ is a *local strong pathwise solution* of (5.1) if

1. $\mathtt{t}$ is an a.s. strictly positive $(\mathscr{F}_t)$-stopping time,
2. the density ϱ is a $\overline{\varrho} + W^{s,2}(\mathbb{R}^3)$-valued $(\mathscr{F}_t)$-progressively measurable process satisfying,
$$\varrho(\cdot \wedge \mathtt{t}) > 0, \quad \varrho(\cdot \wedge \mathtt{t}) \in C\big([0,T]; \overline{\varrho} + W^{s,2}(\mathbb{R}^3)\big), \quad \mathbb{P}\text{-a.s.},$$
3. the velocity field $\mathbf{u}$ is a $W^{s,2}(\mathbb{R}^3)$-valued $(\mathscr{F}_t)$-progressively measurable process satisfying,
$$\mathbf{u}(\cdot \wedge \mathtt{t}) \in C\big([0,T]; W^{s,2}(\mathbb{R}^3)\big), \quad \mathbb{P}\text{-a.s.},$$
4. the following
$$\begin{aligned} \varrho(t \wedge \mathtt{t}) &= \varrho(0) - \int_0^{t\wedge \mathtt{t}} \mathrm{div}(\varrho \mathbf{u})\, \mathrm{d}s, \\ \big(\varrho \mathbf{u}\big)(t \wedge \mathtt{t}) &= \big(\varrho \mathbf{u}\big)(0) - \int_0^{t\wedge \mathtt{t}} \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u})\, \mathrm{d}s \\ &\quad - \int_0^{t\wedge \mathtt{t}} \nabla p(\varrho)\, \mathrm{d}s + \int_0^{t\wedge \mathtt{t}} \Phi(\varrho, \varrho \mathbf{u})\, \mathrm{d}W(s) \end{aligned} \tag{5.16}$$
holds $\mathbb{P}$-a.s. for all $t \in [0,T]$.

In the definition above, we have tacitly assumed that s is large enough in order to provide sufficient regularity for the strong solutions. Classical solutions requires spatial derivatives of ϱ and $\mathbf{u}$ to be continuous $\mathbb{P}$-a.s. This motivates the following definition.

Definition 5.2.8. Fix a stochastic basis with a cylindrical Wiener process and an initial condition as in Definition 5.2.7. We say that $\big(\varrho, \mathbf{u}, (\mathtt{t}_N)_{N\in\mathbb{N}}, \mathtt{t}\big)$ is a *maximal strong pathwise solution* of (5.1) if

1. $(\mathtt{t}_N)_{N\in\mathbb{N}}$ is an increasing sequence of $(\mathscr{F}_t)$-stopping times such that $\mathtt{t}_N < \mathtt{t}$ in the set $[t < T]$, $\lim_{N\to\infty} \mathtt{t}_N = \mathtt{t}$ a.s. and

$$\sup_{t\in[0,\mathtt{t}_N]} \|\mathbf{u}(t)\|_{W^{1,\infty}(\mathbb{R}^3)} \geq N \quad \text{on} \quad [\mathtt{t} < T],$$

2. for each $N \in \mathbb{N}$, the triplet $(\varrho, \mathbf{u}, \mathtt{t}_N)$ is a local strong pathwise solution of (5.1) in the sense of Definition 5.2.7.

The following existence result was recently shown in [17, Theorem 2.4].

Theorem 5.2.9. *Let $s \in \mathbb{N}$ satisfy $s > \frac{3}{2} + 2$ and let $\overline{\varrho} > 0$. Let the coefficients $\mathbf{g}_k$ satisfy hypotheses (5.10)–(5.12) and let $(\varrho_0, \mathbf{u}_0)$ be an $\mathscr{F}_0$-measurable, $\overline{\varrho}+W^{s,2}(\mathbb{R}^3)\times W^{s,2}(\mathbb{R}^3)$-valued random variable such that $\varrho_0 > 0$ $\mathbb{P}$-a.s. Then there exists a unique maximal strong pathwise solution $(\varrho, \mathbf{u}, (\mathtt{t}_N)_{N\in\mathbb{N}}, \mathtt{t})$ to problem (5.1) in the sense of Definition 5.2.8 with the initial condition $(\varrho_0, \mathbf{u}_0)$.*

A crucial observation in the existence theorem, Theorem 5.2.9, is the strict positivity of the density $\varrho > 0$ up to the stopping time $\mathtt{t}_N$. To see this, we observe that the standard maximum principle

$$\begin{aligned} \inf_{\mathbb{R}^3} \varrho_0 \exp\left(- \int_0^{\mathtt{t}_N} \|\mathrm{div}\, \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \mathrm{d}s \right) &\leq \varrho(t,x) \\ &\leq \sup_{\mathbb{R}^3} \varrho_0 \exp\left(\int_0^{\mathtt{t}_N} \|\mathrm{div}\, \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \mathrm{d}s \right) \end{aligned} \tag{5.17}$$

holds $\mathbb{P}$-a.s. for all $(t,x) \in (0, \mathtt{t}_N) \times \mathbb{R}^3$ since $(\varrho, \mathbf{u})$ solves the continuity equation. Consequently, by the definition of the stopping time $\mathtt{t}_N$, the embedding $W^{s,2}(\mathbb{R}^3) \hookrightarrow C^{1,\alpha}(\mathbb{R}^3)$ for $s > \frac{5}{2}$ and some $\alpha > 0$, as well as the assumption on the data $(\varrho_0, \mathbf{u}_0)$, it implies that there exist a constant $c_N = c_N(\overline{\varrho}) > 0$ such that

$$c_N^{-1} \inf_{\mathbb{R}^3} \varrho_0 \leq \varrho(t,x) \leq c_N \sup_{\mathbb{R}^3} \varrho_0 \tag{5.18}$$

holds $\mathbb{P}$-a.s. for all $(t,x) \in (0, \mathtt{t}_N) \times \mathbb{R}^3$. The strict positivity of density thus follow since ϱ_0 is positive $\mathbb{P}$-a.s.

5.2.10 Main result

Our main result reads as follow.

Theorem 5.2.11. *Let $\overline{\varrho} > 0$ be given and suppose that* (5.10)–(5.12) *holds with $s > \frac{3}{2} + 2$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and W, a cylindrical Wiener process on $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that*

$$\left[(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W\right]_{\varepsilon>0} \tag{5.19}$$

is a family of finite energy weak martingale solution to the system (5.4) *in the sense of Definition 5.2.4. On the same stochastic basis $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\big)$, consider the unique maximal strong pathwise solution to the Euler system* (5.1) *in the sense of Definition 5.2.8 given by $\big(\varrho, \mathbf{u}, (\mathtt{t}_N)_{N\in\mathbb{N}}, \mathtt{t}\big)$ and driven by the same cylindrical Wiener process W. Assume that the initial data $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ and $(\varrho_0, \mathbf{u}_0)$ are $\mathscr{F}_0$-measurable and satisfies the assumptions in Section 5.2.1. Then we have*

$$\sup_{t\in(0,T)} \mathbb{E} \int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 + H(\varrho_\varepsilon, \varrho)\right](t \wedge \mathtt{t}_N, \cdot)\, \mathrm{d}x \to 0 \tag{5.20}$$

as $\varepsilon \to 0$ for all $N \in \mathbb{N}$.

Corollary 5.2.12. Suppose that Theorem 5.2.11 hold. Then for all $N \in \mathbb{N}$,

$$\varrho_\varepsilon(t \wedge \mathtt{t}_N, \cdot) \to \varrho(t \wedge \mathtt{t}_N, \cdot) \text{ in } L^{\overline{\gamma}}\big(\Omega \times (0,T); L^{\overline{\gamma}}_{\mathrm{loc}}(\mathbb{R}^3)\big) \tag{5.21}$$

$$(\varrho_\varepsilon \mathbf{u}_\varepsilon)(t \wedge \mathtt{t}_N, \cdot) \to (\varrho\mathbf{u})(t \wedge \mathtt{t}_N, \cdot) \text{ in } L^{\frac{2\overline{\gamma}}{\overline{\gamma}+1}}\big(\Omega \times (0,T); L^{\frac{2\overline{\gamma}}{\overline{\gamma}+1}}_{\mathrm{loc}}(\mathbb{R}^3)\big) \tag{5.22}$$

as $\varepsilon \to 0$ for $\overline{\gamma} = \min\{2, \gamma\}$.

Proof of Corollary 5.2.12. To show (5.21), we first note the inequality

$$H(\varrho, r) \geq c(r) \begin{cases} |\varrho - r|^2 & : \text{ if } r/2 \leq \varrho \leq 2r, \\ 1 + \varrho^\gamma & : \text{ else }, \end{cases} \tag{5.23}$$

c.f. [39, Eq. 4.1]. This yields

$$\begin{aligned}\mathbb{E}\int_0^T\int_K |\varrho_\varepsilon-\varrho|^{\overline{\gamma}}(t\wedge\mathtt{t}_N,\cdot)\,\mathrm{d}x\,\mathrm{d}t &\lesssim \mathbb{E}\int_0^T\int_K H\big(\varrho_\varepsilon,\varrho\big)(t\wedge\mathtt{t}_N,\cdot)\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \mathbb{E}\sup_{t\in(0,T)}\int_{\mathbb{R}^3} H\big(\varrho_\varepsilon,\varrho\big)(t\wedge\mathtt{t}_N,\cdot)\,\mathrm{d}x\end{aligned} \tag{5.24}$$

for all $N\in\mathbb{N}$, for any $t\in[0,T]$ and an arbitrary $K\Subset\mathbb{R}^3$.

Remark 5.2.13. Notice the first inequality in (5.24) follows immediately from (5.23) when $\overline{\gamma}=2$ and when $\overline{\gamma}=\gamma$, we can first use the continuity of

$$L^2((0,T)\times K)\hookrightarrow L^\gamma((0,T)\times K)$$

and then apply (5.23).

Since the first integrand in (5.20) is non-negative, (5.21) follow from (5.20).

Now for (5.22), let us first remark that the identity

$$\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u}=(\varrho_\varepsilon-\varrho)\mathbf{u}+\sqrt{\varrho_\varepsilon}\sqrt{\varrho_\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{u})$$

holds and well as the following Hölder's coefficients

$$\frac{1}{\overline{\gamma}}+\frac{\overline{\gamma}-1}{2\overline{\gamma}}=\frac{1}{2\overline{\gamma}}+\frac{1}{2}=\frac{\overline{\gamma}+1}{2\overline{\gamma}}$$

where in particular $\frac{2\overline{\gamma}}{\overline{\gamma}-1}\leq 6$ and $\frac{6}{5}<\frac{2\overline{\gamma}}{\overline{\gamma}+1}\leq\overline{\gamma}$.

Furthermore, let us set $q:=\frac{2\overline{\gamma}}{\overline{\gamma}+1}$ so that $\frac{q}{2}\leq\frac{\overline{\gamma}}{2}\leq 1$. Also set $L^r_{t,x}:=L^r((0,T)\times K)$ and $L^r_tL^s_x:=L^r(0,T;L^s(K))$ for some $r,s\geq 1$ and $K\Subset\mathbb{R}^3$, then by Hölder inequality, we gain

$$
\begin{aligned}
\mathbb{E}\left\|\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u}\right\|^q_{L^q_{t,x}} &\lesssim_q \mathbb{E}\left\|(\varrho_\varepsilon-\varrho)\mathbf{u}\right\|^q_{L^q_{t,x}}+\mathbb{E}\left\|\sqrt{\varrho_\varepsilon}\sqrt{\varrho_\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{u})\right\|^q_{L^q_{t,x}}\\
&\lesssim \mathbb{E}\left\|\mathbf{u}\right\|^q_{L^{\frac{2\overline{\gamma}}{\overline{\gamma}-1}}_{t,x}}\mathbb{E}\left\|\varrho_\varepsilon-\varrho\right\|^q_{L^{\overline{\gamma}}_{t,x}}+\mathbb{E}\left\|\sqrt{\varrho_\varepsilon}\right\|^q_{L^{2\overline{\gamma}}_{t,x}}\mathbb{E}\left\|\sqrt{\varrho_\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{u})\right\|^q_{L^2_{t,x}}\\
&\lesssim \mathbb{E}\left\|\nabla\mathbf{u}\right\|^q_{L^2_{t,x}}\mathbb{E}\left\|\varrho_\varepsilon-\varrho\right\|^{\overline{\gamma}}_{L^{\overline{\gamma}}_{t,x}}+\mathbb{E}\left\|\varrho_\varepsilon\right\|^{\frac{q}{2}}_{L^{\overline{\gamma}}_{t,x}}\mathbb{E}\left\|\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{u}|^2\right\|^{\frac{q}{2}}_{L^1_{t,x}}\\
&\lesssim \mathbb{E}\left\|\nabla\mathbf{u}\right\|^2_{L^2_{t,x}}\mathbb{E}\left\|\varrho_\varepsilon-\varrho\right\|^{\overline{\gamma}}_{L^{\overline{\gamma}}_{t,x}}+\mathbb{E}\left\|\varrho_\varepsilon\right\|^{\overline{\gamma}}_{L^{\overline{\gamma}}_{t,x}}\mathbb{E}\sup_{t\in(0,T)}\left\|\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{u}|^2\right\|_{L^1_x}.
\end{aligned}
\tag{5.25}
$$

Given that by (5.21) and the previously shown regularity for velocity, recall $(3.87)_1$, the estimate

$$
\mathbb{E}\left\|\nabla\mathbf{u}\right\|^2_{L^2_{t,x}}+\mathbb{E}\left\|\varrho_\varepsilon\right\|^{\overline{\gamma}}_{L^{\overline{\gamma}}_{t,x}}\lesssim 1
$$

holds uniformly in ε, we have gained by further employing (5.20) and (5.21) in (5.25), the following:

$$
\mathbb{E}\left\|(\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u})(t\wedge\mathtt{t}_N,\cdot)\right\|^{\frac{2\overline{\gamma}}{\overline{\gamma}+1}}_{L^{\frac{2\overline{\gamma}}{\overline{\gamma}+1}}((0,T)\times K)}\to 0
\tag{5.26}
$$

as $\varepsilon\to 0$ and thus the claim. □

5.3 Proof of Theorem 5.2.11

To begin the proof of our main theorem in this chapter, we consider

$$
\Big[\big(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P}\big),\varrho_\varepsilon,\mathbf{u}_\varepsilon,W\Big]_{\varepsilon>0},
\tag{5.27}
$$

a family of finite energy weak martingale solution to (5.4), existence of which is guaranteed by Theorem 5.2.5. Our aim is to pass to the limit $\varepsilon\to 0$.

Remark 5.3.1. By [57], any member of the family of weak martingale solution in (5.27) is defined on the standard probability space $\big([0,1],\overline{\mathcal{B}([0,1])},\mathcal{L}\big)$ with a complete right continuous and non-anticipative filtration

$$\sigma\left(\sigma_t[\varrho_\varepsilon]\cup\sigma_t[\mathbf{u}_\varepsilon]\cup\bigcup_{k=1}^{\infty}\sigma_t[\beta_k^\varepsilon]\right),\quad t\in[0,T].$$

So this justify the choice of the same probability space. Without loss of generality, we also assume that they are driven by the same Wiener process.

5.3.2 Application of the relative energy inequality

As mentioned in Section 3.6, the *relative energy inequality* is a tool which enables us to compare a solution or a family of solutions $(\varrho_\varepsilon,\mathbf{u}_\varepsilon)$ of a fluid dynamic system with some smooth comparison functions. In this chapter, these comparison functions will be the unique solution of the Euler system in the sense of Definition 5.2.8 while $(\varrho_\varepsilon,\mathbf{u}_\varepsilon)$ is expected to solve (5.4) in the sense of Definition 5.2.4.

In order to prove Theorem 5.2.11, we recall the pair $(r,\mathbf{U})$ from Section 3.6.1 satisfying

$$(r-\overline{\varrho})\in C_c^\infty\big([0,T]\times\mathbb{R}^3\big),\tag{5.28}$$

$$\mathbf{U}\in C_c^\infty\big([0,T]\times\mathbb{R}^3\big)\tag{5.29}$$

with far-field density $\overline{\varrho}>0$. Given that $C_c^\infty(\mathbb{R}^3)$ is dense in $W^{s,2}(\mathbb{R}^3)$ for $s\in\mathbb{N}$ satisfying $s>3/2+2$, we can approximate the solution $(\varrho,\mathbf{u})$ as given by Theorem 5.2.11, by a sequence of functions $(r,\mathbf{U})$ (not labelled) satisfying (5.28)–(5.29). In order words, we can essentially choose $(r,\mathbf{U})=(\varrho,\mathbf{u})$ in Section 3.6 where $(\varrho,\mathbf{u},(\mathtt{t}_N)_{N\in\mathbb{N}},\mathtt{t})$ is the unique maximal strong pathwise solution to (5.1) which exists by Theorem 5.2.9.

Recall that the stopping time $\mathtt{t}_N$ announces the blow-up and satisfies

$$\sup_{t\in[0,\mathtt{t}_N]}\|\mathbf{u}(t)\|_{W^{1,\infty}(\mathbb{R}^3)}\geq N\quad\text{on}\quad[\mathtt{t}<T];\tag{5.30}$$

Moreover, $(r, \mathbf{U}) = (\varrho, \mathbf{u})$ satisfies an equation of the form (3.235), where

$$D_t^d r = -\mathrm{div}(\varrho \mathbf{u}), \quad \mathbb{D}_t^s r = 0,$$
$$D_t^d \mathbf{U} = -\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\varrho} \nabla p(\varrho), \quad \mathbb{D}_t^s \mathbf{U} = \frac{1}{\varrho} \Phi(\varrho, \varrho \mathbf{u}), \quad \varrho > 0.$$

Recall (5.18).

By Theorem 5.2.9 and (5.10)–(5.12), it is easy to see that (3.238) and (3.239) are satisfied. Furthermore, by an approximation argument for functions satisfying (3.236)–(3.237), we obtain the following for $(\varrho, \mathbf{u})$. In particular, for

$$\mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\right) = \int_{\mathbb{R}^3} \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 + H(\varrho_\varepsilon, \varrho)\right] \mathrm{d}x,$$

we gain from (3.248) that for any $t \in [0, T]$,

$$\begin{aligned} &\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(t) + \varepsilon \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}s \\ &\leq \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(0) + M_{RE}(t) + \int_0^t \mathcal{R}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(s) \, \mathrm{d}s \end{aligned} \tag{5.31}$$

holds $\mathbb{P}$-a.s. where the remainder takes the form

$$\begin{aligned} \mathcal{R}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big) &= \varepsilon \int_{\mathbb{R}^3} \mathbb{S}(\nabla \mathbf{u}) : (\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \varrho_\varepsilon \Big(-\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\varrho} \nabla P(\varrho) + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}\Big) \cdot \big(\mathbf{u} - \mathbf{u}_\varepsilon\big) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \big[-(\varrho - \varrho_\varepsilon) P''(\varrho) \mathrm{div}(\varrho \mathbf{u}) + \nabla P'(\varrho) \cdot (\varrho \mathbf{u} - \varrho_\varepsilon \mathbf{u}_\varepsilon)\big] \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \big[p(\varrho) - p(\varrho_\varepsilon)\big] \mathrm{div}(\mathbf{u}) \, \mathrm{d}x \\ &+ \frac{1}{2} \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \varrho_\varepsilon \left| \frac{\mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)}{\varrho_\varepsilon} - \frac{\mathbf{g}_k(\varrho, \varrho \mathbf{u})}{\varrho} \right|^2 \mathrm{d}x. \end{aligned} \tag{5.32}$$

and the martingale also reduces to

$$\begin{aligned} M_{RE}(t) &= \int_0^t \int_{\mathbb{R}^3} (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon) \, \mathrm{d}x \, \mathrm{d}W \\ &- \int_0^t \int_{\mathbb{R}^3} \varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \frac{\Phi(\varrho, \varrho \mathbf{u})}{\varrho} \, \mathrm{d}x \, \mathrm{d}W. \end{aligned} \tag{5.33}$$

5.3.3 Estimating the residuals

We now wish to control the remainder (5.32) and martingale (5.33) terms appearing on the right-hand side of the energy inequality (5.31). To do this, we first simplify (5.32) by considering the following identities

$$\varrho \nabla P'(\varrho) = \nabla p(\varrho), \quad \varrho \partial_t P'(\varrho) = \partial_t p(\varrho), \quad -\partial_t \varrho = \mathrm{div}(\varrho \mathbf{u})$$

which can be easily verified. Then, it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\frac{\varrho_\varepsilon}{\varrho} \nabla p(\varrho) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) - (\varrho - \varrho_\varepsilon) P''(\varrho) \mathrm{div}(\varrho \mathbf{u}) + \nabla P'(\varrho) \cdot (\varrho \mathbf{u} - \varrho_\varepsilon \mathbf{u}_\varepsilon) \right] \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left[\frac{\varrho_\varepsilon}{\varrho} \nabla p(\varrho) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) + (\varrho - \varrho_\varepsilon) \partial_t \big[P'(\varrho) \big] + \nabla p(\varrho) \cdot \mathbf{u} - \frac{\varrho_\varepsilon}{\varrho} \nabla p(\varrho) \cdot \mathbf{u}_\varepsilon \right] \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left[\partial_t p(\varrho) - \frac{\varrho_\varepsilon}{\varrho} \partial_t p(\varrho) + \nabla p(\varrho) \cdot \mathbf{u} - \frac{\varrho_\varepsilon}{\varrho} \nabla p(\varrho) \cdot \mathbf{u} \right] \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3} \left(\frac{\varrho - \varrho_\varepsilon}{\varrho} \right) \big(\partial_t p(\varrho) + \nabla p(\varrho) \cdot \mathbf{u} \big) \, \mathrm{d}x.
\end{aligned}$$

However, since $(\varrho, \mathbf{u})$ is a strong solution to the continuity equation, it satisfies the *strong* renormalized continuity equation

$$\partial_t p(\varrho) + \nabla p(\varrho) \cdot \mathbf{u} = -\gamma \, p(\varrho) \mathrm{div}(\mathbf{u}).$$

By combining this with the identity $\varrho p'(\varrho) = \gamma p(\varrho)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(\frac{\varrho - \varrho_\varepsilon}{\varrho} \right) \big(\partial_t p(\varrho) + \nabla p(\varrho) \cdot \mathbf{u} \big) \, \mathrm{d}x &= - \int_{\mathbb{R}^3} (\varrho - \varrho_\varepsilon) \frac{\gamma \, p(\varrho)}{\varrho} \, \mathrm{div}(\mathbf{u}) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} (\varrho_\varepsilon - \varrho) p'(\varrho) \, \mathrm{div}(\mathbf{u}) \, \mathrm{d}x.
\end{aligned}$$

Since the solution of the Euler system (5.1) exists up to a stopping time, by collecting the above estimates, we can now deduce from (5.31) that for each $N \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,|\, \varrho, \mathbf{u}\big)(t \wedge \mathtt{t}_N) + \varepsilon \int_0^{t \wedge \mathtt{t}_N} \int_{\mathbb{R}^3} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,|\, \varrho, \mathbf{u}\big)(0) + M_{RE}(t \wedge \mathtt{t}_N) + \int_0^{t \wedge \mathtt{t}_N} \tilde{\mathcal{R}}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,|\, \varrho, \mathbf{u}\big)(s) \, \mathrm{d}s
\end{aligned} \tag{5.34}$$

holds $\mathbb{P}$-a.s. for any $t \in (0,T)$ and each stopping time $\mathtt{t}_N$ and where now,

$$
\begin{aligned}
\tilde{\mathcal{R}}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big) = \varepsilon \int_{\mathbb{R}^3} & \mathbb{S}(\nabla \mathbf{u}) : (\nabla \mathbf{u} - \nabla \mathbf{u}_\varepsilon) \, \mathrm{d}x \\
& + \int_{\mathbb{R}^3} \varrho_\varepsilon \big(\mathbf{u}_\varepsilon - \mathbf{u}\big) \cdot \nabla \mathbf{u} \cdot \big(\mathbf{u} - \mathbf{u}_\varepsilon\big) \, \mathrm{d}x \\
& - \int_{\mathbb{R}^3} \big[p(\varrho_\varepsilon) - (\varrho_\varepsilon - \varrho) p'(\varrho) - p(\varrho)\big] \mathrm{div}(\mathbf{u}) \, \mathrm{d}x \\
& + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^3} \varrho_\varepsilon \left| \frac{\mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)}{\varrho_\varepsilon} - \frac{\mathbf{g}_k(\varrho, \varrho \mathbf{u})}{\varrho} \right|^2 \mathrm{d}x.
\end{aligned}
\tag{5.35}
$$

Now by Hölder inequality and (5.30), we obtain

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3} \varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla \mathbf{u} \cdot \big(\mathbf{u} - \mathbf{u}_\varepsilon\big) \, \mathrm{d}x \right| &\leq \|\nabla \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \, \mathrm{d}x \\
&\leq c(N) \, \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)
\end{aligned}
\tag{5.36}
$$

$\mathbb{P}$-a.s. Since the identity

$$
p(\varrho_\varepsilon) - (\varrho_\varepsilon - \varrho) p'(\varrho) - p(\varrho) = (\gamma - 1) H(\varrho_\varepsilon, \varrho)
$$

holds, it follows that

$$
\begin{aligned}
\bigg| \int_{\mathbb{R}^3} & \big[p(\varrho_\varepsilon) - (\varrho_\varepsilon - \varrho) p'(\varrho) - p(\varrho)\big] \mathrm{div}(\mathbf{u}) \, \mathrm{d}x \bigg| \\
&\leq c \, \|\mathrm{div}\, \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} H(\varrho_\varepsilon, \varrho) \, \mathrm{d}x \\
&\leq c(N) \, \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)
\end{aligned}
\tag{5.37}
$$

$\mathbb{P}$-a.s. And by Young's inequality for bilinear forms and (5.30),

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3} \varepsilon \, \mathbb{S}(\nabla \mathbf{u}) \, : \, (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \, \mathrm{d}x \right| \\
&\leq \frac{1}{2} \varepsilon \int_{\mathbb{R}^3} \Big(\mathbb{S}(\nabla \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla \mathbf{u}) \Big) \, : \, \big(\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}\big) \, \mathrm{d}x + c \, \varepsilon \int_{\mathbb{R}^3} \big| \mathbb{S}(\nabla \mathbf{u}) \big|^2 \, \mathrm{d}x \\
&\leq \frac{1}{2} \varepsilon \int_{\mathbb{R}^3} \Big(\mathbb{S}(\nabla \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla \mathbf{u}) \Big) \, : \, \big(\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}\big) \, \mathrm{d}x + c(N) \, \varepsilon.
\end{aligned}
\tag{5.38}
$$

$\mathbb{P}$-a.s. Finally, we rewrite

$$\begin{aligned}
&\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^3}\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho_\varepsilon}-\frac{\mathbf{g}_k(\varrho,\varrho\mathbf{u})}{\varrho}\right|^2\mathrm{d}x\\
&=\frac{1}{2}\sum_{k\in\mathbb{N}}\int_K\chi_{\{\varrho_\varepsilon\le\varrho/2\}}\,\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho_\varepsilon}-\frac{\mathbf{g}_k(\varrho,\varrho\mathbf{u})}{\varrho}\right|^2\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}\,\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho_\varepsilon}-\frac{\mathbf{g}_k(\varrho,\varrho\mathbf{u})}{\varrho}\right|^2\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_K\chi_{\{\varrho_\varepsilon\ge2\varrho\}}\,\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho_\varepsilon}-\frac{\mathbf{g}_k(\varrho,\varrho\mathbf{u})}{\varrho}\right|^2\mathrm{d}x\\
&=:I_1+I_2+I_3
\end{aligned}\tag{5.39}$$

for $K\Subset\mathbb{R}^3$ (recall (5.12)).

We can now use the inequality $\varrho\le1+\varrho^\gamma$ and (5.23) to conclude that

$$\begin{aligned}
I_1&\le\frac{1}{2}\sum_{k\in\mathbb{N}}\int_K\chi_{\{\varrho_\varepsilon\le\varrho/2\}}\left(\frac{1}{\varrho_\varepsilon}\left|\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)\right|^2+\frac{\varrho_\varepsilon}{\varrho^2}\left|\mathbf{g}_k(\varrho,\varrho\mathbf{u})\right|^2\right)\mathrm{d}x\\
&\le c\int_K\chi_{\{\varrho_\varepsilon\le\varrho/2\}}\big(\varrho_\varepsilon+\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\varrho_\varepsilon|\mathbf{u}|^2\big)\,\mathrm{d}x\\
&\le c(N)\int_K\chi_{\{\varrho_\varepsilon\le\varrho/2\}}\big(1+\varrho_\varepsilon^\gamma+\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{u}|^2\big)\,\mathrm{d}x\\
&\le c(N)\,\mathcal{E}\big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\varrho,\mathbf{u}\big)
\end{aligned}\tag{5.40}$$

$\mathbb{P}$-a.s. For I_2, we use (5.10), (5.11), (5.23), (5.30), the bound on ϱ (i.e. (5.18)) and the mean value theorem to get

$$\begin{aligned}
I_2&\le c\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}\,\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho_\varepsilon}-\frac{\mathbf{g}_k(\varrho,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho}\right|^2\mathrm{d}x\\
&+c\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}\,\varrho_\varepsilon\left|\frac{\mathbf{g}_k(\varrho,\varrho_\varepsilon\mathbf{u}_\varepsilon)}{\varrho}-\frac{\mathbf{g}_k(\varrho,\varrho\mathbf{u})}{\varrho}\right|^2\mathrm{d}x\\
&\le c(N)\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}\Big(|\varrho_\varepsilon-\varrho|^2\big(1+|\varrho_\varepsilon\mathbf{u}_\varepsilon|^2\big)+|\varrho_\varepsilon\mathbf{u}_\varepsilon-\varrho\mathbf{u}|^2\Big)\,\mathrm{d}x\\
&\le c(N)\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}\Big(|\varrho_\varepsilon-\varrho|^2\big(1+|\mathbf{u}|^2\big)+|\varrho_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{u})|^2\Big)\,\mathrm{d}x\\
&\le c(N)\int_K\chi_{\{\varrho/2<\varrho_\varepsilon<2\varrho\}}|\varrho_\varepsilon-\varrho|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{u}|^2\,\mathrm{d}x\\
&\le c(N)\,\mathcal{E}\big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\varrho,\mathbf{u}\big)
\end{aligned}\tag{5.41}$$

$\mathbb{P}$-a.s.

Again, by the use of the inequality $\varrho \leq 1 + \varrho^\gamma$ and (5.23), we gain

$$
\begin{aligned}
I_3 &\leq \frac{1}{2} \sum_{k\in\mathbb{N}} \int_K \chi_{\{\varrho_\varepsilon > 2\varrho\}} \left(\frac{1}{\varrho_\varepsilon} \left|\mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\right|^2 + \frac{\varrho_\varepsilon}{\varrho^2} \left|\mathbf{g}_k(\varrho, \varrho\mathbf{u})\right|^2 \right) \mathrm{d}x \\
&\leq c \int_K \chi_{\{\varrho_\varepsilon > 2\varrho\}} \big(\varrho_\varepsilon + \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon |\mathbf{u}|^2 \big) \,\mathrm{d}x \\
&\leq c \int_K \chi_{\{\varrho_\varepsilon > 2\varrho\}} \big(\varrho_\varepsilon + \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 + \varrho_\varepsilon |\mathbf{u}|^2 \big) \,\mathrm{d}x \\
&\leq c \int_K \chi_{\{\varrho_\varepsilon > 2\varrho\}} \Big(\varrho_\varepsilon \big(1 + |\mathbf{u}|^2\big) + \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \Big) \,\mathrm{d}x \\
&\leq c(N) \int_K \chi_{\{\varrho_\varepsilon > 2\varrho\}} \big(1 + \varrho_\varepsilon^\gamma + \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}|^2 \big) \,\mathrm{d}x \\
&\leq c(N) \, \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)
\end{aligned}
\tag{5.42}
$$

$\mathbb{P}$-a.s. So by combining the estimates (5.40)–(5.42), we can conclude from (5.39) that

$$
\frac{1}{2} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^3} \varrho_\varepsilon \left| \frac{\mathbf{g}_k(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)}{\varrho_\varepsilon} - \frac{\mathbf{g}_k(\varrho, \varrho\mathbf{u})}{\varrho} \right|^2 \mathrm{d}x \leq c(N) \, \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big). \tag{5.43}
$$

5.3.4 Conclusion

Collecting the estimates (5.36)–(5.43), we have shown that

$$
\int_0^{t\wedge \mathtt{t}_N} \mathcal{R}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big) \,\mathrm{d}s \leq c(N) \left(\int_0^{t\wedge \mathtt{t}_N} \mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(s) \,\mathrm{d}s + \varepsilon \right) \tag{5.44}
$$

For the martingale (5.33), since M_{RE} starts at zero (recall that by definition, $W(0) = 0$), and martingales are constant on average, it follows that for any $t \in (0, T)$,

$$
\mathbb{E}\, M_{RE}(t \wedge \mathtt{t}_N) = 0 \tag{5.45}
$$

By combining (5.44) and (5.45), we can deduce from (5.34) and Gronwall's lemma that

$$
\begin{aligned}
\sup_{t\in(0,T)} \mathbb{E} \Big[\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(t \wedge \mathtt{t}_N) \Big] + \varepsilon\, \mathbb{E} \bigg[\int_0^{t\wedge \mathtt{t}_N} \int_{\mathbb{R}^3} \big(\mathbb{S}(\nabla \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla \mathbf{u})\big) \\
: (\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}s \bigg] \leq c(N)\, \mathbb{E} \Big[\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(0) + \varepsilon \Big].
\end{aligned}
\tag{5.46}
$$

Now we note that we have

$$\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(0) = \int_{\mathbb{R}^3} \left[\frac{1}{2}\varrho_{0,\varepsilon}\big|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\big|^2 + H(\varepsilon\varrho_{0,\varepsilon}, \varrho_0)\right] \mathrm{d}x \tag{5.47}$$

which converges to zero in expectation by the assumptions in Section 5.2.1. Consequently, we obtain

$$\mathcal{E}\big(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid \varrho, \mathbf{u}\big)(0) \to 0 \tag{5.48}$$

as $\varepsilon \to 0$. The convergence (5.20) then follow by passing to the limit $\varepsilon \to 0$ in (5.46).

Chapter 6

A multi-scale limit of a randomly forced rotating 3-D fluid

6.1 Introduction

In large scale geophysical fluid dynamics like in the study of the oceans and the atmosphere, inertial forces like the centrifugal force, gravitational force and Coriolis force play a crucial role in the evolution of fluids. In these situations, it is not enough to represent all external forces as one function, say $\mathbf{f}$, since each one of these forces provides a peculiar characteristics to the fluids evolution.

Also in large scale, phenomena like lunar tides which is caused by gravitational forces and to an extent centrifugal forces, as well as heat, leads to fluid profiles where lighter fluids lies on top of heavier ones with differing densities. This phenomenon is referred to as *density stratification* or simply *(fluid) stratification*. This stratification inhibits vertical fluid motion which are parallel to the gravitational force and as such these large scale fluid motion becomes essentially horizontal. Further information on these phenomenon can be found in [89].

Additionally, one may also incorporate a stochastic forcing term to account for turbulence. By combining the contributions of all these forces, we obtain the stochastic compressible Navier–Stokes system for rotating fluids.

Our aim is to study a singular limit problem of the following system for rotating fluids

$$
\begin{aligned}
&\mathrm{d}\varrho + \operatorname{div}(\varrho \mathbf{u})\mathrm{d}t = 0, \\
&\mathrm{d}(\varrho \mathbf{u}) + \Big[\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ro}}\varrho(\mathbf{e}_3 \times \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla p(\varrho)\Big]\mathrm{d}t \\
&\qquad\qquad = \operatorname{div} \mathbb{S}(\nabla \mathbf{u})\,\mathrm{d}t + \frac{1}{\mathrm{Fr}^2}\varrho \nabla G + \Phi(\varrho, \varrho \mathbf{u})\mathrm{d}W
\end{aligned}
\tag{6.1}
$$

where the density ϱ and velocity vector field $\mathbf{u}$ takes its values from the space $\mathcal{O} = \mathbb{R}^2 \times (0,1)$. The term $\frac{1}{\mathrm{Ro}}\varrho(\mathbf{e}_3 \times \mathbf{u})$ in (6.1) above accounts for rotation in the fluid due to Coriolis forces, $\mathbf{e}_3 = (0,0,1)$ is the unit vector in the vertical x_3-direction and the factor $\frac{1}{\mathrm{Ro}}$ - which is the reciprocal of the Rossby number - measures the intensity or the speed of this rotation. When the Rossby number is small, the balance of forces is highly influenced by the Coriolis force leading to a predominantly horizontal fluid profile.

The centrifugal force term is essentially of the form $\nabla G \approx \nabla(|x_1|^2 + |x_2|^2)$ with $(x_1, x_2) \in \mathbb{R}^2$ and with $\frac{1}{\mathrm{Fr}^2}$ - the squared reciprocal of the Froude number - quantifying the level of stratification in the fluid. Here, $p(\varrho) = \varrho^\gamma$ with $\gamma > \frac{3}{2}$ is the isentropic pressure, Ma is the Mach number and the viscous stress tensor is

$$
\mathbb{S}(\nabla \mathbf{u}) := \nu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3}\operatorname{div} \mathbf{u}\mathbb{I} \right) \tag{6.2}
$$

with viscosity coefficients satisfying $\nu > 0$.

Remark 6.1.1. Notice that unlike the previous chapters, we have excluded the so-called 'bulk' part of the viscous stress tensor (6.2). This is for technical reasons which is made clear in the proof of (6.49).

A prototype for the stochastic forcing term will be

$$
\Phi(\varrho, \varrho \mathbf{u})\,\mathrm{d}W \approx \varrho\,\mathrm{d}W^1 + \varrho \mathbf{u}\,\mathrm{d}W^2 \tag{6.3}
$$

for a pair of identically distributed independent Wiener processes W^1 and W^2. We give the precise assumptions on the noise term in Section 6.2.2.

If we set the Rossby number $\mathrm{Ro} = \varepsilon$, the Froude number $\mathrm{Fr} = \varepsilon$ and the Mach number $\mathrm{Ma} = \varepsilon^m$ for some $m \gg 1$, then given a sequence $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ of *finite energy weak martingale solutions* to (6.1) (see Definition 6.2.7 for the precise definition), we show that its limit $\mathbf{U} = [\mathbf{U}_h(x_1, x_2), 0]$, solves the 2-dimensional Navier–Stokes system

$$\begin{aligned} \mathrm{d}\mathbf{U}_h + [\mathrm{div}_h(\mathbf{U}_h \otimes \mathbf{U}_h) + \nabla_h \pi - \nu \Delta_h \mathbf{U}_h]\, \mathrm{d}t = \mathcal{P}\Phi(1, \mathbf{U}_h)\, \mathrm{d}W, \\ \mathrm{div}_h \mathbf{U}_h = 0. \end{aligned} \tag{6.4}$$

Here, π is an associated pressure term, $\mathcal{P}$ represents Helmoltz decomposition onto solenoidal vector fields and the subscript h which stands for 'horizontal', represents the first two component of a 3-D vector. The precise statement of this result is given in Theorem 6.2.14.

The two dimensionality of (6.4) follow from the stratification effect of fast rotation corresponding to when the Rossby and Froude numbers becomes small, whereas the incompressibility of the system is as a result of smallness of the Mach number.

The additional Coriolis term has a regularizing effect on the system. In particular, the convergence leading to (6.4) is obtained in *probability* which is stronger than the *convergence in law* obtained in the purely incompressible limit result studied in [11] on the torus and on $\mathbb{R}^3$ in Chapter 4 above. To summarize, fast rotation due to Coriolis force leads to the 2-dimensional system (6.4) for which uniqueness result is available. We state this uniqueness result in Theorem 6.2.12. Consequently, we gain strong convergence in probability as a result of uniqueness.

In the deterministic setting, the analysis of incompressible rotating fluids have been studied by several authors. For an extensive review and introduction on the topic, the reader might want to see [19]. However, important contributions include the works of Babin et al [3, 4] where they study certain class of solutions to the system using amongst other techniques, the Littlewood–Paley dyadic decomposition and further tools from algebraic geometry.

More recent work include [85], where the authors study a deterministic homogeneous compressible inviscid system on the whole space $\mathbb{R}^3$. They analyse the system under fast rotations with isotropic scale corresponding to when $\mathrm{Ro} = \mathrm{Ma} = \varepsilon$. This involved decomposing the system into a linear part and nonlinear part. Using Strichartz-type estimates, they establish the convergence to zero for the linear part. The non-linear part is then analysed using bootstrapping methods and some harmonic analysis tools including paradifferential calculus.

In [37], the authors study the deterministic counterpart of this limit problem from a 3-dimensional compressible Navier–Stokes system to a 2-dimensional incompressible Navier–Stokes system when $\mathrm{Ro} = \mathrm{Ma} = \varepsilon \to 0$ and with no centrifugal force effect, i.e. $\mathrm{Fr} = \infty$, $\frac{1}{\infty} := 0$. Using the so called RAGE theorem, they established the convergence to zero of the acoustic energy. The subsequent limit system is then given as a *stream function* for the incompressible 2-D Navier–Stokes system.

For a more general scalings of the form $\mathrm{Ro} = \varepsilon$, $\mathrm{Ma} = \varepsilon^m$ where $m \geq 1$ and $\mathrm{Fr} = \varepsilon$, which is more in line with what we study in this chapter, the authors in [36] then study the limit problem under the influence of centrifugal force. If $m = 1$, they obtain a 2-D linear system with radially symmetric solutions whereas the multi-scale limit problem corresponding to $m \gg 1$ converges to the 2-D Navier–Stokes system. In this later case, the choice of m subsequently eliminates the effect due to the centrifugal force.

There is very little results for stochastic problems involving rotation. In [50], they study averaging results for the 3-D stochastic incompressible Navier–Stokes system under fast rotation on a periodic domain. Here, an additive white noise is considered and the limit variables solves the so-called 3-D stochastic resonant averaged equation.

As far as we can tell, there are no available results for compressible rotating fluids with stochastic forcing. Indeed, apart from the low-Mach number result in [11] and in Chapter 4 above, the other result pertaining to such singular limits, we believe, is contained in [12]. In [12], the combined effect of the low Mach number regime and the high Reynolds number Re is studied on a torus. From a 3-D stochastic compressible Navier–Stokes system (without rotation), they obtain in the limit, a

3-D stochastic incompressible Euler equation.

We now give a brief outline of this Chapter. First of all, unless otherwise stated, the assumptions that we make in Section 6.2 will apply throughout the chapter. We also define in that section, the various concept of solutions in Section 6.2.6, as well as state the main result in Section 6.2.13.

We will then devote the entirety of Section 6.3 and Section 6.4 to the proof of our main Theorem 6.2.14. Our compactness arguments in Section 6.3.9 will start by first establishing uniform estimates in Section 6.3.1. Obtaining such uniform bounds will rely on the relative energy inequality introduced in Section 3.6 in the context of stochastic compressible fluids on the whole space. We then show in Lemma 6.3.10, tightness of the joint law on the path space defined on uniformly bounded sequences obtained in Section 6.3.1. Finally, we conclude the section by using the Jakubowski–Skorokhod theorem , Theorem 2.4.29 to establish almost sure convergent subsequences in the topology of the path space mentioned above.

Section 6.3.13 through to Section 6.5.1 will involve the justification of the limit system by treating the most important terms separately. A crucial part of the analysis involves the corresponding acoustic wave equation which we study in Section 6.4.1. In this regard, the proof of the crucial result, Lemma 6.4.6, will rely on Fourier analysis, semigroup theory and regularization to obtain the mild form of the acoustic system. We then use Strichartz-type estimates to obtain uniform bounds for the (rescaled) gradient part of momentum. By scaling back, we eventually show that this part of the momentum vanishes in the limit. We then follow this by showing in Section 6.4.8 that the vertical average of the solenoidal part of momentum converges in the limit, to the full velocity.

In Section 6.3.7 we show that by considering the vertical averages, one can conclude that the Coriolis term is a gradient vector field and thus weakly solenoidal. Also, as mentioned in the previous paragraph, we study in Section 6.4.8, just the vertical average of the solenoidal part of momentum. This leads us to justify in Section 6.4.10 that any residual or oscillatory term obtained after the taking of vertical averages does not contribute to the limit system. We will then devote Section 6.5.1 to the proof of Lemma 6.5.2 which identifies the limit in the nonlinear convective term.

Finally, we complete the proof of Theorem 6.2.14 in Section 6.5.3 by using the unique (pathwise) solvability of the limit problem (6.4). The main tool is based on the recent result by Breit et al [15, Theorem 2.10.3] that extends the original Gyöngy–Krylov's characterization of convergence in probability on Polish spaces [55] to quasi-Polish spaces (this includes Banach spaces with weak topology). Having established convergence in law Section 6.3 and with 2-D uniqueness Theorem 6.2.12 in hand, we gain convergence in probability to the limit problem.

6.2 Preliminaries

6.2.1 Notations and definitions

Let us start with a few notations pertinent to this chapter. We set $Q_T = (0,T) \times \mathcal{O}$ for fixed $T > 0$ and consider the following microscopic state variables. For $x = (x_1, x_2, x_3) \in \mathcal{O}$, we let $x_h = (x_1, x_2) \in \mathbb{R}^2$ represent its first two or 'horizontal' component and with the third or 'vertical' component $x_3 \in (0,1)$. We now define on $\mathcal{O}$, the macroscopic state variables $\varrho = \varrho(t,x)$ and $\mathbf{u} = \mathbf{u}(t,x)$ which are respectively, a non-negative scaler and a three dimensional Euclidean vector valued functions representing the density and velocity fields. The vector valued function $(\varrho\mathbf{u}) = (\varrho\mathbf{u})(t,x)$ represents the momentum.

We shall reserve the following short-hand notation $\|\cdot\|_{L^p_x}$ for the globally defined norms on the whole space $L^p(\mathbb{R}^3)$. In this case, the integral over the whole space $\mathbb{R}^3$ is to be understood as the extension by zeroes outside of $\mathcal{O}$ whenever the function is only defined on $\mathcal{O}$. An extension of this notation will be $\|\cdot\|_{L^p_{t,x}}$ or $\|\cdot\|_{L^q_\omega L^p_{t,x}}$ which will refer to the norms on $L^p((0,T) \times \mathbb{R}^3)$ and $L^q(\Omega; L^p(0,T) \times \mathbb{R}^3))$ respectively, as well as similar variants. However, integrals over proper subsets of $\mathbb{R}^3$ will be made explicit. For example, we shall write $\|\cdot\|_{L^p(K)}$ for the usual Lebesgue norm whenever

$K \subset \mathbb{R}^3$.

Here and throughout the rest of this chapter, we shall use the notations $\langle u, v\rangle = \int_{\mathcal{O}} uv \,\mathrm{d}x$ and $\langle u, v\rangle_h = \int_{\mathbb{R}^2} uv \,\mathrm{d}x_h$ in 3-D and 2-D respectively and where the 2-D version corresponds to the horizontal or first two Cartesian components of the 3-D version.

6.2.2 Assumptions on the stochastic force

Set $\mathbf{m} := \varrho\mathbf{u}$ and assume that there exists a compact set $\mathcal{K} \subset \mathbb{R}^2$ for which we set $K := \mathcal{K} \times [0,1] \subset \overline{\mathcal{O}}$. We then assume the existence of some C^1-functions $\mathbf{g}_k$: $\mathcal{O} \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ whose decompositions are made up of functions $\underline{\mathbf{g}}_k : \mathcal{O} \times \mathbb{R}_+ \to \mathbb{R}$ and $\alpha_k := \alpha_k(x) : \mathcal{O} \to \mathbb{R}$ such that

$$\mathbf{g}_k(x, \varrho, \mathbf{m}) = \underline{\mathbf{g}}_k(x, \varrho) + \alpha_k(x)\,\mathbf{m}, \quad k \in \mathbb{N}. \tag{6.5}$$

Equation (6.5) is motivated by a similar choice in [11] but our coefficients are assumed to satisfy the uniform bounds

$$\sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty, \quad \sum_{k\in\mathbb{N}} \left|\underline{\mathbf{g}}_k(x,\varrho)\right|^2 \lesssim \varrho^2, \quad \sum_{k\in\mathbb{N}} \left|\nabla_\varrho\, \underline{\mathbf{g}}_k(x,\varrho)\right|^2 \lesssim 1. \tag{6.6}$$

Then if we define the map $\Phi(\varrho, \mathbf{m}) : \mathfrak{U} \to L^1(K)$ by

$$\Phi(\varrho, \mathbf{m})e_k = \mathbf{g}_k(\cdot, \varrho(\cdot), \mathbf{m}(\cdot)) = \underline{\mathbf{g}}_k(\cdot, \varrho(\cdot)) + \alpha_k(\cdot)\,\mathbf{m}(\cdot), \tag{6.7}$$

where

$$\mathrm{spt}(\mathbf{g}_k) \Subset K, \quad \text{for any } k \in \mathbb{N}, \tag{6.8}$$

we can use the embedding $L^1(K) \hookrightarrow W^{-l,2}(K)$ where $l > \frac{3}{2}$, and (6.6)–(6.8) to show that

$$\begin{aligned}
\|\Phi(\varrho, \mathbf{m})\|^2_{L_2(\mathfrak{U}; W^{-l,2}(\mathcal{O}))} &= \sum_{k \in \mathbb{N}} \|\mathbf{g}_k(x, \varrho, \mathbf{m})\|^2_{W^{-l,2}(\mathcal{O})} \\
&= \sum_{k \in \mathbb{N}} \|\mathbf{g}_k(x, \varrho, \mathbf{m})\|^2_{W^{-l,2}(K)} \lesssim \sum_{k \in \mathbb{N}} \|\mathbf{g}_k(x, \varrho, \mathbf{m})\|^2_{L^1(K)} \\
&\lesssim (\varrho)_K \int_K \sum_{k \in \mathbb{N}} \varrho^{-1} |\mathbf{g}_k(x, \varrho, \mathbf{m})|^2 \, \mathrm{d}x \\
&\lesssim (\varrho)_K \int_K \sum_{k \in \mathbb{N}} \left(\varrho^{-1} |\underline{\mathbf{g}}_k(x, \varrho)|^2 + |\alpha_k|^2 \, \varrho \, |\mathbf{u}|^2 \right) \mathrm{d}x \\
&\lesssim (\varrho)_K \int_K \left(1 + \varrho^\gamma + \varrho |\mathbf{u}|^2 \right) \mathrm{d}x.
\end{aligned} \tag{6.9}$$

where $(\varrho)_K$ represents the average density over the compact set K and where we have used $\varrho \leq 1 + \varrho^\gamma$ in the last step. The left-hand side of (6.9) is therefore uniformly bounded provided $\varrho \in L^\gamma_{\mathrm{loc}}(\mathcal{O})$ and $\sqrt{\varrho}\mathbf{u} \in L^2_{\mathrm{loc}}(\mathcal{O})$. If so, then the stochastic integral $\int_0^\cdot \Phi(\varrho, \mathbf{m}) \mathrm{d}W$ is a well-defined $(\mathscr{F}_t)$-martingale taking value in $W^{-l,2}(\mathcal{O})$.

As already mentioned in the introduction, we expect in the limit, a process that solves the 2-D Navier–Stokes system, Eq. (6.4). Consequently, it follows from (6.8)–(6.9) that the limit diffusion coefficient is of the kind

$$\Psi(\mathbf{U}_h) e_k = \mathbf{g}_{k,h}\big(\cdot, \mathbf{U}_h(\cdot)\big) = \underline{\mathbf{g}}_{k,h}(\cdot) + \alpha_k(\cdot) \, \mathbf{U}_h(\cdot)$$

with $\Psi(\mathbf{U}_h) = \mathcal{P}\Phi(1, \mathbf{U}_h(x_h))$ and coefficients satisfying the bound

$$\sum_{k \in \mathbb{N}} |\alpha_k|^2 + \sum_{k \in \mathbb{N}} \left| \underline{\mathbf{g}}_{k,h}(x_h, 1) \right|^2 \lesssim 1. \tag{6.10}$$

Subsequently, the estimate for the noise term in Eq. (6.4) becomes:

$$\begin{aligned}
\|\Psi(\mathbf{U}_h)\|^2_{L_2(\mathfrak{U}; W^{-l,2}(\mathbb{R}^2))} &\lesssim \sum_{k \in \mathbb{N}} \|\underline{\mathbf{g}}_{k,h}(x_h) + \alpha_k(x_h) \, \mathbf{U}_h\|^2_{L^1(\mathcal{K})} \\
&\lesssim \int_{\mathcal{K}} \sum_{k \in \mathbb{N}} \left(|\underline{\mathbf{g}}_{k,h}(x_h)|^2 + |\alpha_k \, \mathbf{U}_h|^2 \right) \mathrm{d}x \\
&\lesssim \int_{\mathcal{K}} (1 + |\mathbf{U}_h|^2) \mathrm{d}x
\end{aligned} \tag{6.11}$$

where $\mathcal{K}$ is the same compact set hidden in (6.8) above and where we have used the continuity of the operator $\mathcal{P}$.

Lastly, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ as in (3.8) such that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt.

6.2.3 Boundary and far field conditions

Since we are working on an semi bounded/unbounded spatial domain, we supplement our system (6.1) with the far field condition

$$\varrho \to \overline{\varrho}_\varepsilon, \quad \mathbf{u} \to 0 \quad \text{as} \quad |x_h| \to \infty, \quad \mathbb{P}\text{-a.s.} \tag{6.12}$$

for some time independent function $\overline{\varrho}_\varepsilon = \overline{\varrho}_\varepsilon(x) > 0$ as well as the complete slip boundary condition for the velocity field

$$\begin{aligned} \mathbf{u}\cdot\mathbf{n}\big|_{\partial\mathcal{O}} &= \pm\, u_3\big|_{\partial\mathcal{O}} = 0, \\ \left([\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}\right)\big|_{\partial\mathcal{O}} &= (S_{23}, -S_{13}, 0)\big|_{\partial\mathcal{O}} = 0 \end{aligned} \tag{6.13}$$

where $\mathbf{n} = (0, 0, \pm 1)$ is the outer normal vector to the boundary; so as to entirely eliminate the influence of boundary effects.

6.2.4 The relative energy functional

We now introduce the relative energy functional which compares 'solutions' of (6.1) with some smooth functions r and $\mathbf{U}$. Let start by first defining the following.

For the *isentropic* pressure function $p(z) = z^\gamma$ with $p \in C^1[0,\infty) \cap C^2(0,\infty)$,

$$P(\varrho) = \varrho \int_1^\varrho z^{\gamma-2}\,\mathrm{d}z \tag{6.14}$$

represent the corresponding *pressure potential.* C.f. (3.17).

Now we assume that the (smooth) functions $r, \mathbf{U}$ are random variables that are adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ and satisfies:

$$r > 0, \quad (r - \overline{\varrho}_\varepsilon) \in C_c^\infty([0,T] \times \overline{\mathcal{O}}), \quad \mathbf{U} \in C_c^\infty([0,T] \times \mathcal{O}), \tag{6.15}$$

$\mathbb{P}$-a.s. Additionally, we assume that $\overline{\varrho}_\varepsilon = \overline{\varrho}_\varepsilon(x)$ solves the static problem

$$\nabla \overline{\varrho}_\varepsilon^\gamma = \varepsilon^{2(m-1)} \overline{\varrho}_\varepsilon \nabla G \quad \text{in} \quad \mathcal{O} \tag{6.16}$$

for a non-negative time independent deterministic force G that satisfy

$$G = G(x) \geq 0, \quad G \in W^{1,1}(\mathcal{O}) \cap W^{1,\infty}(\mathcal{O}). \tag{6.17}$$

We now set

$$H(\varrho, r) = P(\varrho) - P'(r)(\varrho - r) - P(r) \tag{6.18}$$

and define

$$\mathcal{E}\left(\varrho, \mathbf{u} \,|\, r, \mathbf{U}\right)(t, \cdot) := \int_{\mathcal{O}} \left[\frac{\varrho}{2}|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\mathrm{Ma}^2} H(\varrho, r)\right](t, \cdot)\,\mathrm{d}x \tag{6.19}$$

to be the *relative energy functional.*

Remark 6.2.5. By using the identity (6.14), one can easily check that (6.16) is equivalent to solving $\nabla P'(\overline{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} \nabla G$ so that

$$P'\big(\overline{\varrho}_\varepsilon(x)\big) = \varepsilon^{2(m-1)} G(x) + P'(1). \tag{6.20}$$

Since G is non-negative, it follows from (6.20) that for any $x \in \mathcal{O}$ and $m > 1$,

$$1 \leq \overline{\varrho}_\varepsilon(x) < c \quad \text{and} \quad \overline{\varrho}_\varepsilon(x) \to 1$$

as $\varepsilon \to 0$ for some $c > 0$. Furthermore, by using the Lipschitz continuity of G, we can deduce from (6.20) that for any $x \in \mathcal{O}$ with $|x| \leq k\varepsilon^{-2\alpha}$, $k > 0$ and $0 \leq \alpha \leq m-1$,

$$|\overline{\varrho}_\varepsilon(x) - 1| \lesssim \varepsilon^{2(m-1-\alpha)} \tag{6.21}$$

with the constant $c > 0$ depending only on $k > 0$.

6.2.6 Concepts of solution

We now define different notions of solution that will be considered in this chapter. We follow the concept on the whole space, see Section 3.2.4, where we now define on our special geometry $\mathcal{O}$, a corresponding solution to (6.1) which is weak in both the probabilistic and PDE sense. This is given in Definition 6.2.7 below and is analogous to Definition 3.2.6 for non-rotating fluids.

Definition 6.2.7. Let $\overline{\varrho} > 0$. If Λ is a Borel probability measure on $L^\gamma_{\mathrm{loc}}(\mathcal{O}) \times L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathcal{O})$, then we say that $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho, \mathbf{u}, W]$ is a *finite energy weak martingale solution* of equation (6.1) with initial law Λ provided

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;

2. W is a $(\mathscr{F}_t)$-cylindrical Wiener process;

3. the density ϱ satisfies $\varrho \geq 0$, $t \mapsto \langle \varrho(t, \cdot), \phi \rangle \in C([0,T])$ for any $\phi \in C^\infty_c(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho(t, \cdot), \phi \rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|^p_{L^\gamma(K)}\right] < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

4. the velocity field $\mathbf{u}$ is an $(\mathscr{F}_t)$-adapted random distribution and

$$\mathbb{E}\left[\int_0^T \|\mathbf{u}\|^2_{W^{1,2}(K)}\,\mathrm{d}t\right]^p < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$,

5. the momentum $\varrho\mathbf{u}$ satisfies $t \mapsto \langle \varrho\mathbf{u}, \boldsymbol{\varphi}\rangle \in C([0,T])$ for any $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \varrho\mathbf{u}, \boldsymbol{\phi}\rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\varrho\mathbf{u}\|^p_{L^{\frac{2\gamma}{\gamma+1}}(K)}\right] < \infty$$

for all $1 \leq p < \infty$ and all $K \Subset \mathbb{R}^3$;

6. there exists $\mathscr{F}_0$-measurable random variables $(\varrho_0, \varrho_0\mathbf{u}_0) = (\varrho(0), \varrho\mathbf{u}(0))$ such that $\Lambda = \mathbb{P} \circ (\varrho_0, \varrho_0\mathbf{u}_0)^{-1}$;

7. for all $\psi \in C_c^\infty(\mathcal{O})$ and $\boldsymbol{\phi} \in C_c^\infty(\mathcal{O})$ and all $t \in [0,T]$, it holds $\mathbb{P}$-a.s.

$$\begin{aligned}
&\langle \varrho(t)\,,\, \psi\rangle = \langle \varrho_0, \psi\rangle + \int_0^t \langle \varrho\mathbf{u}, \nabla\psi\rangle \mathrm{d}r,\\
&\langle \varrho\mathbf{u}(t)\,,\, \boldsymbol{\phi}\rangle = \langle \varrho_0\mathbf{u}_0\,,\, \boldsymbol{\phi}\rangle + \int_0^t \langle \varrho\mathbf{u}\otimes\mathbf{u}, \nabla\boldsymbol{\phi}\rangle \mathrm{d}r - \frac{1}{\mathrm{Ro}}\int_0^t \langle \varrho(\mathbf{e}_3 \times \mathbf{u})\,,\, \boldsymbol{\phi}\rangle \mathrm{d}r\\
&\quad - \int_0^t \langle \mathbb{S}(\nabla\mathbf{u})\,,\, \mathrm{div}\boldsymbol{\phi}\rangle \mathrm{d}r + \frac{1}{\mathrm{Ma}^2}\int_0^t \langle p(\varrho)\,,\, \mathrm{div}\boldsymbol{\phi}\rangle \mathrm{d}r\\
&\quad + \frac{1}{\mathrm{Fr}^2}\int_0^t \langle \varrho\,\nabla G\,,\, \boldsymbol{\phi}\rangle \mathrm{d}r + \int_0^t \langle \Phi(\varrho, \varrho\mathbf{u})\mathrm{d}W\,,\, \boldsymbol{\phi}\rangle;
\end{aligned}$$

8. the following inequality

$$\begin{aligned}
&\int_{\mathcal{O}} \left[\frac{\varrho}{2}|\mathbf{u}|^2 + \frac{H(\varrho, \overline{\varrho})}{\mathrm{Ma}^2}\right](\tau)\,\mathrm{d}x + \int_0^\tau \int_{\mathcal{O}} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \leq \left(\int_{\mathcal{O}} \left[\frac{|\varrho_0\mathbf{u}_0|^2}{2\varrho_0} + \frac{H(\varrho_0, \overline{\varrho})}{\mathrm{Ma}^2}\right]\mathrm{d}x\right)\\
&\quad + \frac{1}{2}\int_0^\tau \left(\int_{\mathcal{O}} \sum_{k\in\mathbb{N}} \varrho^{-1}|\mathbf{g}_k(x, \varrho, \varrho\mathbf{u})|^2\,\mathrm{d}x\right)\mathrm{d}t + M_R(\tau)
\end{aligned} \tag{6.22}$$

holds $\mathbb{P}$-a.s. for a real-valued martingale M_R given by

$$M_R(\tau) = \int_0^\tau \int_{\mathcal{O}} \mathbf{u}\cdot\Phi(\varrho, \varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}W, \tag{6.23}$$

and satisfying the estimate

$$\mathbb{E}\left(\sup_{t\in[0,T]} |M_R|^p\right) \lesssim_p 1 + \mathbb{E}\left[\int_{\mathcal{O}} \left(\frac{|\varrho_0\mathbf{u}_0|^2}{2\varrho_0} + \frac{H(\varrho_0, \overline{\varrho})}{\mathrm{Ma}^2}\right)\mathrm{d}x\right]^p. \tag{6.24}$$

for all $p \in [1, \infty)$.

9. In addition, $(6.1)_1$ holds in the renormalized sense. That is, for any $\phi \in \mathcal{D}'(\mathcal{O})$ and $b \in C^0[0,\infty) \cap C^1(0,\infty)$ such that $|b'(t)| \leq ct^{-\lambda_0}$, $t \in (0,1]$, $\lambda_0 < 1$ and $|b'(t)| \leq ct^{\lambda_1}$, $t \geq 1$ where $c > 0$ and $-1 < \lambda_1 < \infty$, we have that

$$\mathrm{d}\langle b(\varrho), \phi\rangle = \langle b(\varrho)\mathbf{u}, \nabla\phi\rangle \mathrm{d}t - \langle (b(\varrho) - b'(\varrho)\varrho) \operatorname{div}\mathbf{u}, \phi\rangle \mathrm{d}t. \tag{6.25}$$

Remark 6.2.8. Following a similar argument as in the proof of Theorem 3.2.12, one can establish on $\mathbb{T}^2_L \times \mathbb{T}_1$ instead of $\mathbb{T}^3_L$, the existence of a finite energy weak martingale solution to (6.1) in the sense of Definition 6.2.7 under the assumption that (6.8), (6.6), (6.26), (6.27) and (6.28) hold. Here, $\mathbb{T}^p_L = \big([-L,L]|_{\{L,L\}}\big)^p$ and $\mathbb{T}_1 = [-1,1]|_{\{-1,1\}}$ are the p-D and 1-D flat tori with periods $2L \geq 1$ and 2 respectively. For L fixed, existence of a finite energy weak martingale solution follows from [16]. The aim will then be to pass to the limit as $L \to \infty$ in analogy to the proof of Theorem 3.2.12. However this will yield a result posed on $\mathcal{O} = \mathbb{R}^2 \times \mathbb{T}_1$ instead of the original geometry $\mathcal{O} = \mathbb{R}^2 \times (0,1)$. This aforementioned reformulation into a purely periodic problem with a corresponding boundary condition is allowed after a special symmetrization of our density and velocity vector fields as given in [36, Eq. 1.7]. This was originally proposed in [32] and has already been applied in the stochastic setting [14]. For completeness, we state them below.

$$\begin{aligned} \varrho\left(\cdot\,, x_h, -x_3\right) = \varrho\left(\cdot\,, x_h, x_3\right), \quad \mathbf{u}_h\left(\cdot\,, x_h, -x_3\right) &= \mathbf{u}_h\left(\cdot\,, x_h, x_3\right), \\ -u_3\left(\cdot\,, x_h, -x_3\right) &= u_3\left(\cdot\,, x_h, x_3\right). \end{aligned} \tag{6.26}$$

That is, the horizontal component of velocity and density are extended from $(0,1)$ to $\mathbb{T}_1$ as an even function in x_3 whereas the vertical component of velocity is extended to an odd function in x_3.

In accordance with (6.26), the functions $\mathbf{g}_k(x, \varrho, \mathbf{m})$ in (6.8) are assumed to satisfy

$$\begin{aligned} -g_{k,3}\left(x_h, -x_3, \cdot, \mathbf{m}_h, -m_3\right) &= g_{k,3}\left(x_h, x_3, \cdot, \mathbf{m}_h, m_3,\right), \\ \mathbf{g}_{k,h}\left(x_h, -x_3, \cdot, \mathbf{m}_h, -m_3\right) &= \mathbf{g}_{k,h}\left(x_h, x_3, \cdot, \mathbf{m}_h, m_3\right), \end{aligned} \tag{6.27}$$

where $g_{k,3}$ and $\mathbf{g}_{k,h}$, agrees correspondingly, to the 'vertical' and 'horizontal' components of the noise term. Lastly, we extend the potential of the centrifugal force to $\mathbb{T}_1$ as an even function in x_3 , i.e.,

$$G(x_h, -x_3) = G(x_h, x_3). \tag{6.28}$$

These symmetric assumptions (6.26), (6.27) and (6.28) are thus, implicitly implied throughout the rest of this chapter.

Remark 6.2.9. Notice that the term $\varrho(\mathbf{e}_3 \times \mathbf{u})$ is orthogonal to $\mathbf{u}$ and so it vanishes during the compactness argument when we test the momentum equation with the velocity. Also, the centrifugal forcing term is easily controlled by a similar estimate as in (6.39) below.

The nature of the limit system (6.4) naturally leads to a corresponding definition of a solution in 2-D. Typically, this can either be simultaneously weak in the probabilistic and PDE sense, in analogy to Definition 6.2.7 above, or strong in at least one of these senses. The former notion is stated in Definition 6.2.10 below. For the later, which we state in Definition 6.2.11, the solutions are weak in the PDE sense but strong in the sense of probability. This follows from uniqueness in 2-D which is currently unavailable for the 3-D counterpart. In this later case, the underlying probability space is fixed in advance. Consequently, existence of solution in the sense of the later yields the former. However, the analysis involved in this chapter is such that, both versions are required.

Definition 6.2.10. Let Λ be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{R}^2)$. Then we say that $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{u}, W]$ is a *weak martingale solution* of equation (6.4) with initial datum Λ provided:

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
2. W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
3. $\mathbf{u}$ is $(\mathscr{F}_t)$-adapted, $\mathbf{u} \in C_w\left([0,T]; L^2_{\mathrm{div}}(\mathbb{R}^2)\right) \cap L^2(0,T; W^{1,2}_{\mathrm{div}}(\mathbb{R}^2))$ $\mathbb{P}$-a.s. and

for all $p \in [1, \infty)$,

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\mathbf{u}\|^2_{L^2(\mathbb{R}^2)}\right]^p + \mathbb{E}\left[\left(\int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\mathbb{R}^2)} \mathrm{d}t\right)^p\right] < \infty, \tag{6.29}$$

4. $\Lambda = \mathbb{P} \circ (\mathbf{u}(0))^{-1}$,

5. for all $\boldsymbol{\phi} \in C^\infty_{c,\mathrm{div}}(\mathbb{R}^2)$ and all $t \in [0, T]$, it holds $\mathbb{P}$-a.s.,

$$\begin{aligned}\langle \mathbf{u}(t)\,,\,\boldsymbol{\phi}\rangle_h = \langle \mathbf{u}(0)\,,\,\boldsymbol{\phi}\rangle_h - \int_0^t \langle \mathbf{u}\otimes\mathbf{u}\,,\,\nabla\boldsymbol{\phi}\rangle_h\,\mathrm{d}r + \nu\int_0^t \langle \nabla\mathbf{u}\,,\,\boldsymbol{\phi}\rangle_h\,\mathrm{d}r \\ + \int_0^t \langle \mathcal{P}\Phi(1,\mathbf{u})\mathrm{d}W\,,\,\boldsymbol{\phi}\rangle_h\end{aligned} \tag{6.30}$$

where $\mathcal{P}$ is the Helmholtz decomposition onto the space of solenoidal vector fields.

An even stronger notion of solution for the incompressible stochastic Navier–Stokes system is the concept of *weak pathwise solution* given below.

Definition 6.2.11. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis with an $(\mathscr{F}_t)$- cylindrical Wiener process W. Let $\mathbf{u}_0$ be an $\mathscr{F}_0$-measurable random variable. Then we say that $\mathbf{u}$ is a *weak pathwise solution* of the Navier–Stokes system (6.4) with initial datum $\mathbf{u}_0$ provided:

1. the velocity $\mathbf{u}$ is $(\mathscr{F}_t)$-adapted, $\mathbf{u} \in C_w\left([0, T]; L^2_{\mathrm{div}}(\mathbb{R}^2)\right) \cap L^2(0, T; W^{1,2}_{\mathrm{div}}(\mathbb{R}^2))$ $\mathbb{P}$-a.s. and for all $p \in [1, \infty)$, (6.29) holds true,

2. the equality $\mathbf{u}(0) = \mathbf{u}_0$ holds $\mathbb{P}$-a.s.,

3. for all $\phi \in C^\infty_{c,\mathrm{div}}(\mathbb{R}^2)$ and all $t \in [0, T]$, Eq. (6.30) holds $\mathbb{P}$-a.s.

Theorem 6.2.12. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis with an $(\mathscr{F}_t)$-cylindrical Wiener process W and let $\mathbf{U}_0$ be an $\mathscr{F}_0$-measurable random variable belonging to the space $L^p(\Omega; L^2_{\mathrm{div}}(\mathbb{R}^2))$ for all $p \in [1, \infty)$. If* (6.11) *holds, then there exists a unique weak pathwise solution to* (6.4) *in the sense of Definition 6.2.11 with initial condition $\mathbf{U}_0$.*

Proof. See [79] with $\mathfrak{U} = l^2(\mathbb{R}^2)$. ∎

6.2.13 Main result

We now state the main result of this chapter. Theorem 6.2.14 below corresponds to the simultaneous low Rossby - low Mach - low Froude number limit result of the stochastic compressible Navier–Stokes–Coriolis system taking into account, the influence of centrifugal force.

Theorem 6.2.14. *Set* $\mathrm{Ma} = \varepsilon^m$, *for* $m > 10$ *and* $\mathrm{Ro} = \mathrm{Fr} = \varepsilon$ *in* (6.1)*. Let* $\gamma > \frac{3}{2}$ *and assume that* $\mathbf{U}_0 = [\mathbf{U}_{h,0}, 0] \in L^2(\mathbb{R}^3)$*. Consider initial data* $\big(\varrho_{0,\varepsilon}, (\varrho\mathbf{u})_{0,\varepsilon}\big) \in L^\gamma(\mathcal{O}) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{O})$ *satisfying*

$$
\begin{aligned}
&\varrho_{0,\varepsilon} = \overline{\varrho}_\varepsilon + \varepsilon^m \varrho^{(1)}_{0,\varepsilon} > 0, \quad \big\{\sqrt{\overline{\varrho}_\varepsilon}\, \mathbf{u}_{0,\varepsilon}\big\}_{\varepsilon>0} \in L^2(\mathcal{O}),\\
&\big\{\overline{\varrho}_\varepsilon^{\frac{\gamma-2}{2}}\, \varrho^{(1)}_{0,\varepsilon}\big\}_{\varepsilon>0} \in L^2(\mathcal{O}), \quad \big\{\varrho^{(1)}_{0,\varepsilon}\big\}_{\varepsilon>0} \in L^\infty \cap L^1(\mathcal{O}),\\
&|(\varrho\mathbf{u})_{0,\varepsilon} - \mathbf{U}_0| + |\varrho_{0,\varepsilon} - \overline{\varrho}_\varepsilon| \leq \varepsilon^m M,
\end{aligned}
\tag{6.31}
$$

for a constant $M > 0$ *and for* $\overline{\varrho}_\varepsilon > 0$ *solving* (6.16)*. If the collection* $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W]$ *is a family of finite energy weak martingale solution of* (6.1) *in the sense of Definition 6.2.7 with initial law* $\Lambda_\varepsilon = \mathbb{P} \circ [\varrho_{0,\varepsilon}, (\varrho\mathbf{u})_{0,\varepsilon}]^{-1}$, $\varepsilon \in (0,1)$ *and uniformly bounded moment estimate*

$$
\int_{L^\gamma_x \times L_x^{\frac{2\gamma}{\gamma+1}}} \left\| \frac{1}{2}\frac{|\varrho\mathbf{u}|^2}{\varrho} + \frac{1}{\varepsilon^{2m}} H\left(\varrho, \overline{\varrho}_\varepsilon\right) \right\|_{L^1_x}^p \mathrm{d}\Lambda_\varepsilon(\varrho, \varrho\mathbf{u}) \lesssim 1
\tag{6.32}
$$

for all $p \in [1, \infty)$ *and for a constant* $c = c(p) > 0$ *independent of* ε*, then*

$$
\begin{aligned}
\varrho_\varepsilon \to 1 \quad &\textit{in} \quad L^\infty(0,T; L^{\min\{2,\gamma\}}_{\mathrm{loc}}(\mathcal{O})),\\
\mathbf{u}_\varepsilon \to \mathbf{U} \quad &\textit{in} \quad \big(L^2(0,T; W^{1,2}(\mathcal{O})), w\big),
\end{aligned}
\tag{6.33}
$$

in probability and where $\mathbf{U} = [\mathbf{U}_h(t, x_h), 0]$ *is the unique weak pathwise solution of* (6.4) *in the sense of Definition 6.2.11 with the initial condition* $\mathbf{U}_0 = \mathbf{U}_{h,0}(x_h)$*.*

Remark 6.2.15. Similar to the explanation contained in the paragraph after (3.32), we consider the family $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W]_{\varepsilon>0}$ rather than $[(\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t), \mathbb{P}^\varepsilon); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon]_{\varepsilon>0}$ in Theorem 6.2.14 above. This is because the subsequent application of stochastic compactness argument due to Jakubowski will yield respectively, the existence of a probability space or some probability spaces. However, whatever the case may be (either the space is singular or plural), it is shown by Jakubowski that it (they) can be considered as *standard probability space* $\big([0,1], \overline{\mathcal{B}([0,1])}, \mathcal{L}\big)$. So although the theorem will hold for $(\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t), \mathbb{P}^\varepsilon)$, without loss of generality, it is enough to consider $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$. The same loss of generality justifies considering W rather than W_ε.

6.3 Uniform estimates and compactness arguments

This section is devoted to preparations towards the proof of our main theorem. We start be establishing a dissipative estimate for the energy of the compressible system (6.1).

6.3.1 Relative energy inequality and uniform bounds

Since the collection $[(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}); \varrho_\varepsilon, \mathbf{u}_\varepsilon, W]$ is a sequence of finite energy weak martingale solution of (6.1), we can deduce from (6.22) that

$$\begin{aligned}\int_{\mathcal{O}} \left[\frac{\varrho_\varepsilon}{2}|\mathbf{u}_\varepsilon|^2 + \frac{H\big(\varrho_\varepsilon, \overline{\varrho}_\varepsilon\big)}{\varepsilon^{2m}}\right](\tau)\,\mathrm{d}x + \int_0^\tau \int_{\mathcal{O}} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ \leq \int_{\mathcal{O}} \left[\frac{\varrho_{\varepsilon,0}|\mathbf{u}_{\varepsilon,0}|^2}{2} + \frac{H\big(\varrho_{\varepsilon,0}, \overline{\varrho}_\varepsilon\big)}{\varepsilon^{2m}}\right]\mathrm{d}x \\ + \frac{1}{2}\int_0^\tau \int_{\mathcal{O}} \sum_{k\in\mathbb{N}} \varrho_\varepsilon^{-1} |\mathbf{g}_k(x, \varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x\,\mathrm{d}t + M_R^\varepsilon(\tau)\end{aligned} \tag{6.34}$$

holds $\mathbb{P}$-a.s. with M_R^ε given by

$$M_R^\varepsilon(\tau) = \int_0^\tau \int_{\mathcal{O}} \mathbf{u}_\varepsilon \cdot \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}W. \tag{6.35}$$

By the mass compatibility condition,

$$\int_{\mathcal{O}} \big(\varrho_\varepsilon - \overline{\varrho}_\varepsilon\big)\,\mathrm{d}x = 0, \tag{6.36}$$

it follows from (6.6)–(6.9) that

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_0^\tau\int_{\mathcal{O}}\frac{\varrho_\varepsilon^{-1}}{2}|\mathbf{g}_k(x,\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim \int_0^\tau\int_K\frac{1}{2}\big(\varrho_\varepsilon+\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\big)\mathrm{d}x\,\mathrm{d}t \\ \lesssim \int_0^\tau\int_{\mathcal{O}}\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\mathcal{O}}\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim_\tau \int_0^\tau\int_{\mathcal{O}}\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{O}}\overline{\varrho}_\varepsilon\,\mathrm{d}x.\end{aligned} \tag{6.37}$$

Finally, we give a formal clarification of the apparent loss of the deterministic forcing term. We first recall from Section 3.6 that one applies Itô's formula to certain functionals in order to derive the relative energy inequality. This is similar to testing the momentum balance equation with the velocity vector so that in the case of rotating fluids (6.1), one expects that the following term

$$\int_0^\tau\int_{\mathcal{O}}\frac{1}{\varepsilon^2}\varrho_\varepsilon\nabla G\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \tag{6.38}$$

appears. However we note that

$$\begin{aligned}\int_0^\tau\int_{\mathcal{O}}\frac{1}{\varepsilon^2}\varrho_\varepsilon\nabla G\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t &= -\int_{\mathcal{O}}\int_0^\tau\frac{G}{\varepsilon^2}\mathrm{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{\mathcal{O}}\frac{P'(\overline{\varrho}_\varepsilon)}{\varepsilon^{2m}}\big(\varrho_\varepsilon(\tau)-\varrho_{\varepsilon,0}\big)\,\mathrm{d}x \\ &= \int_{\mathcal{O}}\frac{P'(\overline{\varrho}_\varepsilon)}{\varepsilon^{2m}}\big(\varrho_\varepsilon(\tau)-\overline{\varrho}_\varepsilon\big)\,\mathrm{d}x-\int_{\mathcal{O}}\frac{P'(\overline{\varrho}_\varepsilon)}{\varepsilon^{2m}}\big(\varrho_{\varepsilon,0}-\overline{\varrho}_\varepsilon\big)\,\mathrm{d}x\end{aligned} \tag{6.39}$$

where we have used the continuity equation and (6.20). However, the right-hand terms in (6.39) are precisely, the first order Taylor expansion terms hidden in $H(\varrho_\varepsilon,\overline{\varrho}_\varepsilon)$ and $H(\varrho_{\varepsilon,0},\overline{\varrho}_\varepsilon)$ respectively so that in fact, information given by G is captured in (6.34).

Now by applying Gronwall's lemma, we can combine (6.34), and (6.37) to get

$$\begin{aligned}&\sup_{t\in(0,T)}\mathcal{E}\left(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\overline{\varrho}_\varepsilon,\mathbf{0}\right)(t,x)+\int_0^T\int_{\mathcal{O}}\mathbb{S}(\nabla\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim 1+\mathcal{E}\left(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\overline{\varrho}_\varepsilon,\mathbf{0}\right)(0,x)+\sup_{t\in(0,T)}|M_R^\varepsilon|\end{aligned} \tag{6.40}$$

$\mathbb{P}$-a.s. By invoking the estimate (6.24), we get by taking p-th moments in (6.40) that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in(0,T)}\int_{\mathcal{O}}\frac{\varrho_\varepsilon}{2}|\mathbf{u}_\varepsilon|^2(t,\cdot)\,\mathrm{d}x\right]^p+\mathbb{E}\left[\sup_{t\in(0,T)}\int_{\mathcal{O}}\frac{1}{\varepsilon^{2m}}H(\varrho_\varepsilon,\overline{\varrho}_\varepsilon)(t,\cdot)\,\mathrm{d}x\right]^p\\&+\mathbb{E}\left[\int_0^T\int_{\mathcal{O}}\mathbb{S}(\nabla\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right]^p\lesssim\mathbb{E}\big[1+\mathcal{E}\left(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\overline{\varrho}_\varepsilon,\mathbf{0}\right)(0)\big]^p.\end{aligned}\tag{6.41}$$

Finally, we observe that for any such $p\in[1,\infty)$, the inequality

$$\begin{aligned}&\mathbb{E}\big[1+\mathcal{E}\left(\varrho_\varepsilon,\mathbf{u}_\varepsilon\mid\overline{\varrho}_\varepsilon,\mathbf{0}\right)(0,x)\big]^p\\&\quad\le c(p)\left[1+\mathbb{E}\left(\int_{\mathcal{O}}\left[\frac{|\varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}|^2}{2\varrho_{\varepsilon,0}}+\frac{1}{\varepsilon^{2m}}H\left(\varrho_{\varepsilon,0},\overline{\varrho}_\varepsilon\right)\right]\mathrm{d}x\right)^p\right]\\&\quad=c(p)\left[1+\int_{L^\gamma_x\times L^{\frac{2\gamma}{\gamma+1}}_x}\left\|\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}+\frac{1}{\varepsilon^{2m}}H\left(\varrho,\overline{\varrho}_\varepsilon\right)\right\|^p_{L^1_x}\mathrm{d}\Lambda_\varepsilon(\varrho,\mathbf{m})\right]\end{aligned}$$

holds. Subsequently, from the boundedness assumption on the initial law (i.e., the moment estimate (6.32)), we can conclude from (6.41) that for any $p\in[0,\infty)$,

$$\mathbb{E}\left(\sup_{t\in(0,T)}\int_{\mathcal{O}}\frac{1}{\varepsilon^{2m}}\,H(\varrho_\varepsilon,\overline{\varrho}_\varepsilon)\,\mathrm{d}x\right)^p\le c_p\tag{6.42}$$

and

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in(0,T)}\int_{\mathcal{O}}\frac{\varrho_\varepsilon}{2}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\right)^p&\le c_p,\\\mathbb{E}\left(\int_{Q_T}\mathbb{S}(\nabla\mathbf{u}_\varepsilon):\nabla\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right)^p&\le c_p\end{aligned}\tag{6.43}$$

uniformly in ε.

In the following, unless otherwise specified, we shall always refer to 'balls' as 3-D objects of the form

$$B_k:=\big\{x\in\mathcal{O}\,:\,|x_h|\le k\big\}.\tag{6.44}$$

Also, we follow [42, Page 144] and define $\mathscr{O}_{\mathrm{ess}}$ and $\mathscr{O}_{\mathrm{res}}$ to be fixed subsets of $(0,\infty)$

defined by

$$
\begin{aligned}
\mathscr{O}_{\text{ess}} &:= \{\varrho \in (0,\infty) \,:\, \overline{\varrho}/2 < \varrho < 2\overline{\varrho}\} \\
&= \{\varrho \in (0,\infty) \,:\, |\varrho - 5\overline{\varrho}/4| < 3\overline{\varrho}/4\}, \\
\mathscr{O}_{\text{res}} &:= (0,\infty) \setminus \mathscr{O}_{\text{ess}}
\end{aligned}
$$

respectively. Then for fixed $\omega \in \Omega$, we define the measurable subsets of $\Omega \times (0,T) \times \mathcal{O}$ by

$$
\begin{aligned}
\mathcal{M}^{\varepsilon}_{\text{ess}} &:= \{(\omega,t,x) \in \Omega \times (0,T) \times \mathcal{O} \,:\, \varrho_\varepsilon(\omega,t,x) \in \mathscr{O}_{\text{ess}}\}, \\
\mathcal{M}^{\varepsilon}_{\text{res}} &:= \big(\Omega \times (0,T) \times \mathcal{O}\big) \setminus \mathcal{M}^{\varepsilon}_{\text{ess}}
\end{aligned}
$$

respectively. Subsequently, the decomposition of an integrable function h on the random time-space cylinder into its *essential* and *residual* parts, i.e.,

$$
h = [h]_{\text{ess}} + [h]_{\text{res}}, \quad \text{where} \quad [h]_{\text{ess}} = h\mathbb{1}_{\mathcal{M}^{\varepsilon}_{\text{ess}}},
$$

holds. Given the above definitions, we can now show the following lemma.

Lemma 6.3.2. Let $m > 1 + \alpha$, then

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in(0,T)} \int_{B_{k\varepsilon^{-\alpha}}} \left[\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon^m}\right]^2_{\text{ess}} \mathrm{d}x\right)^p &\leq c_{p,k}, \\
\mathbb{E}\left(\sup_{t\in(0,T)} \int_{B_{k\varepsilon^{-\alpha}}} \left[1 + \varrho_\varepsilon^\gamma\right]_{\text{res}} \mathrm{d}x\right)^p &\leq \varepsilon^{2m} c_{p,k}
\end{aligned}
\tag{6.45}
$$

for balls $B_{k\varepsilon^{-\alpha}} \subset \overline{\mathcal{O}}$ of radius $k\varepsilon^{-\alpha} > 0$.

Proof. First of all, we note that

$$
\mathbb{1}_{\{|\varrho_\varepsilon - 5\overline{\varrho}_\varepsilon/4| < 3\overline{\varrho}_\varepsilon/4\}} \Leftrightarrow \mathbb{1}_{\{\overline{\varrho}_\varepsilon/2 \leq \varrho_\varepsilon < 2\overline{\varrho}_\varepsilon\}}.
$$

So we can deduce from (5.23) that

$$\big|\varrho_\varepsilon - \overline{\varrho}_\varepsilon\big|^2_{\text{ess}} + \big[1 + \varrho_\varepsilon^\gamma\big]_{\text{res}} \lesssim H(\varrho_\varepsilon, \overline{\varrho}_\varepsilon), \tag{6.46}$$

Recall (6.18).

So from (6.42),

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in(0,T)} \int_{B_{k\varepsilon^{-\alpha}}} \left[\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon^m}\right]^2_{\text{ess}} \mathrm{d}x\right)^p \\ \leq &\mathbb{E}\left(\sup_{t\in(0,T)} \int_{\mathcal{O}} \left[\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon^m}\right]^2_{\text{ess}} \mathrm{d}x\right)^p \leq c_{p,k} \end{aligned} \tag{6.47}$$

and similarly for $(6.3.2)_2$. □

Remark 6.3.3. Notice that the proof of Lemma 6.3.2 is independent of the hypothesis $m > 1 + \alpha$ so long as we consider a subset of $\mathcal{O}$. However, the choice of balls $B_{k\varepsilon^{-\alpha}}$ satisfying this condition will be made clear in Section 6.4.1 below.

Lemma 6.3.4. For all $p \in [1, \infty)$, we have

$$\mathbb{E}\left(\sup_{t\in(0,T)} \int_{B_{k\varepsilon^{-\alpha}}} \frac{1}{\varepsilon^{2m}} \big[\varrho_\varepsilon^\gamma - \gamma(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - \overline{\varrho}_\varepsilon^\gamma\big] \,\mathrm{d}x\right)^p \leq (1 + \varepsilon^{2(m-1-\alpha)}) c_{p,k},$$

uniformly in ε.

Proof. We first notice that

$$\begin{aligned} \varrho_\varepsilon^\gamma - \gamma(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - \overline{\varrho}_\varepsilon^\gamma &= \big[\varrho_\varepsilon^\gamma - \gamma\overline{\varrho}_\varepsilon^{\gamma-1}(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - \overline{\varrho}_\varepsilon^\gamma\big] \\ &\quad + \gamma(\overline{\varrho}_\varepsilon^{\gamma-1} - 1)(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) \\ &= (\gamma - 1)H(\varrho_\varepsilon, \overline{\varrho}_\varepsilon) + \gamma(\overline{\varrho}_\varepsilon^{\gamma-1} - 1)(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) \end{aligned} \tag{6.48}$$

where for $x \in B_{k\varepsilon^{-\alpha}}$, we get from (6.21) that

$$\big|\overline{\varrho}_\varepsilon(x)^{\gamma-1} - 1\big| \leq c\,\big|\overline{\varrho}_\varepsilon(x) - 1\big| \leq c\,\varepsilon^{2(m-1-\alpha)}.$$

The claim then follows from (6.42) and Lemma 6.3.2. □

Lemma 6.3.5. For all $p \in [1, \infty)$ and ball $B \subset \mathcal{O}$, we have that

$$\mathbb{E} \left| \left(\int_0^T \left\| \mathbf{u}_\varepsilon \right\|_{W^{1,2}(\mathcal{O})}^2 \mathrm{d}t \right)^{\frac{1}{2}} \right|^p \lesssim 1, \tag{6.49}$$

$$\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \right\|_{L^2(\mathcal{O})} \right|^p \lesssim 1, \tag{6.50}$$

$$\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon^m} \right\|_{L^{\min\{2,\gamma\}}(B)} \right|^p \lesssim 1, \tag{6.51}$$

and

$$\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \varrho_\varepsilon \right\|_{L^\gamma(B)} \right|^p \lesssim 1, \tag{6.52}$$

$$\mathbb{E} \left| \sup_{t \in [0,T]} \left\| \varrho_\varepsilon \mathbf{u}_\varepsilon \right\|_{L^{\frac{2\gamma}{\gamma+1}}(B)} \right|^p \lesssim 1, \tag{6.53}$$

$$\mathbb{E} \left| \left(\int_0^T \left\| \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right\|_{L^{\frac{6\gamma}{4\gamma+3}}(B)}^2 \mathrm{d}t \right)^{\frac{1}{2}} \right|^p \lesssim 1, \tag{6.54}$$

uniformly in ε.

Proof. The first two follows immediately from (6.43) and a version of Korn's inequality [42, Theorem 10.17], c.f. [36, Eq. 2.20]. The bound (6.51) follows from Lemma 6.3.2 and (6.21). The last three (6.52)–(6.54) can be found in (3.49) and (3.54). □

Lemma 6.3.6. For all $p \in [1, \infty)$, we have that

$$\varrho_\varepsilon \to 1 \text{ in } L^p(\Omega; L^\infty(0, T; L^{\min\{2,\gamma\}}_{\mathrm{loc}}(\mathcal{O})))$$

as $\varepsilon \to 0$.

Proof. This is a direct consequence of (6.51) and (6.21). □

Now let set $\mathrm{Ma} = \varepsilon^m$, $\mathrm{Ro} = \mathrm{Fr} = \varepsilon$. Then we observe that by setting $r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon^m}$, [1]

[1] This quantity is sometimes referred to as the density fluctuation.

we derive from Eq. (6.1) the following:

$$\begin{aligned} &\varepsilon^m \mathrm{d} r_\varepsilon + \mathrm{div}\,(\varrho_\varepsilon \mathbf{u}_\varepsilon)\,\mathrm{d}t = 0, \\ \varepsilon^m \mathrm{d}\,(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \big[\varepsilon^{m-1}(\mathbf{e}_3 \times \varrho_\varepsilon \mathbf{u}_\varepsilon) + \gamma \nabla r_\varepsilon\big]\mathrm{d}t &= \varepsilon^m \mathbf{F}_\varepsilon\,\mathrm{d}t \\ &\quad + \varepsilon^{2(m-1)} r_\varepsilon \nabla G\,\mathrm{d}t + \varepsilon^m \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon)\mathrm{d}W \end{aligned} \tag{6.55}$$

in the sense of distributions and where we have used (6.16),

$$\mathbf{F}_\varepsilon := \mathrm{div}\,(\mathbb{S}(\nabla \mathbf{u}_\varepsilon)) - \mathrm{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon^{2m}} \nabla[\varrho_\varepsilon^\gamma - \gamma(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - \overline{\varrho}_\varepsilon^\gamma] \tag{6.56}$$

and for any $K \Subset \mathcal{O}$

$$\mathbf{F}_\varepsilon \in L^p\left(\Omega; L^2\left(0, T; W^{-l,2}(K)\right)\right) \tag{6.57}$$

uniformly in ε for $l > 5/2$. The uniform estimate (6.57) follows from Lemma 6.3.4, (6.49) and (6.54) and is similar to the proof of (4.37).

Finally, one can also infer from (6.51) and (6.17) that

$$r_\varepsilon \nabla G \in L^p(\Omega; L^\infty(0, T; L^{\min\{2,\gamma\}}_{\mathrm{loc}}(\mathcal{O}))). \tag{6.58}$$

6.3.7 Analysis of the Coriolis term

We wish to show in this section that the Coriolis term is a gradient vector field provided we consider its vertical average. cf. [36, Section 3] and [51, Section 3]. To see this, let first consider the following notation:

$$\ulcorner g \urcorner = \fint_{\mathbb{T}_1} g\,\mathrm{d}x_3 = \frac{1}{|\mathbb{T}_1|} \int_{\mathbb{T}_1} g\,\mathrm{d}x_3 \tag{6.59}$$

for any function g defined on $\mathcal{O}$. Then we observes that if we set $\mathbf{Y}_\varepsilon := \mathcal{P}\,(\varrho_\varepsilon \mathbf{u}_\varepsilon)$, we have that $\mathrm{div}(\ulcorner \mathbf{Y}_\varepsilon \urcorner) = \partial_{x_1} \ulcorner Y^1_\varepsilon \urcorner + \partial_{x_2} \ulcorner Y^2_\varepsilon \urcorner = 0$. As such,

$$\mathrm{curl}\,(\mathbf{e}_3 \times \ulcorner \mathbf{Y}_\varepsilon \urcorner) = \left(0, 0, \partial_{x_1} \ulcorner Y^1_\varepsilon \urcorner + \partial_{x_2} \ulcorner Y^2_\varepsilon \urcorner\right) = \mathbf{0}. \tag{6.60}$$

Remark 6.3.8. It is crucial at this point to consider the vertical average of the solenoidal part of momentum since otherwise, the curl of the Coriolis term for the full momentum isn't zero. This taking of the vertical average is a reason why we require the special geometry $\mathcal{O}$ rather than the whole space $\mathbb{R}^3$.

We also observe that for any potential Ψ, the following identity

$$\ulcorner \mathbf{e}_3 \times \nabla\Psi \urcorner = \mathbf{e}_3 \times \ulcorner \nabla\Psi \urcorner = \mathbf{e}_3 \times \nabla \ulcorner \Psi \urcorner \tag{6.61}$$

holds. Subsequently, we will use any of the identities in (6.61) interchangeably throughout the rest of this chapter.

6.3.9 Compactness

As in [36, Eq. 3.3], we introduce the smooth family of cut-off functions η_ε satisfying

$$\begin{aligned} &\eta_\varepsilon \in C_c^\infty(\mathbb{R}^2), \quad 0 \leq \eta_\varepsilon \leq 1, \quad \eta_\varepsilon(x_h) \equiv 1 \text{ in } B_{\varepsilon^{-\alpha}}, \\ &\eta_\varepsilon(x_h) = 0 \text{ if } |x_h| \geq 2\varepsilon^{-\alpha}, \quad \left|\nabla \eta_\varepsilon(x_h)\right| \leq 2\varepsilon^{\alpha} \text{ for } x_h \in \mathbb{R}^2 \end{aligned} \tag{6.62}$$

where since $m > 10$, we can choose α in (6.62) such that

$$1 + \frac{m}{2} < \alpha < \frac{3m}{4} - \frac{3}{2}. \tag{6.63}$$

To explore compactness, let define the following spaces:

$$\begin{aligned} \chi_{\ulcorner \varrho \mathbf{u} \urcorner} &= C_w \left([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{\text{loc}}(\mathcal{O}) \right), \\ \chi_{\mathbf{u}} &= \left(L^2(0,T; W^{1,2}(\mathcal{O})), w \right), \\ \chi_{\varrho} &= C_w \left([0,T]; L^{\gamma}_{\text{loc}}(\mathcal{O}) \right), \\ \chi_W &= C\left([0,T]; \mathfrak{U}_0 \right), \end{aligned}$$

and let

1. $\mu_{\ulcorner \mathcal{P}(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon) \urcorner}$ be the law of $\ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \urcorner$ on the space $\chi_{\ulcorner \varrho \mathbf{u} \urcorner}$,

2. $\mu_{\mathbf{u}_\varepsilon}$ be the law of $\mathbf{u}_\varepsilon$ on $\chi_{\mathbf{u}}$,

3. $\mu_{\varrho_\varepsilon}$ be the law of ϱ_ε on the space χ_ϱ,

4. μ_W be the law of W on the space χ_W.

Now we let $\mu^{\varepsilon,\delta}$ and $\nu^{\varepsilon,\delta}$ be the joint laws of

$$(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \urcorner, \varrho_\delta, \mathbf{u}_\delta, \ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\delta \mathbf{u}_\delta\right) \urcorner)$$

and

$$(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \urcorner, \varrho_\delta, \mathbf{u}_\delta, \ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\delta \mathbf{u}_\delta\right) \urcorner, W)$$

respectively on the path space $\chi = \chi_\varrho \times \chi_{\mathbf{u}} \times \chi_{\ulcorner \varrho\mathbf{u} \urcorner} \times \chi_\varrho \times \chi_{\mathbf{u}} \times \chi_{\ulcorner \varrho\mathbf{u} \urcorner}$ and $\chi^J = \chi \times \chi_W$ respectively.

Lemma 6.3.10. The collection $\{\mu_{\ulcorner \mathcal{P}(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon) \urcorner}; \, \varepsilon \in (0,1)\}$ is tight on $\chi_{\ulcorner \varrho\mathbf{u} \urcorner}$.

Proof. We have shown in Section 6.3.7 that the vertical average of the Coriolis term is curl-free meaning that it is a gradient vector. We now combine the approach of [11, Proposition 3.6] and [42, Section 5.4.2] and consider the projection of (6.55) onto solenoidal fields. This is done by the special choice of divergence-free test function $\mathcal{P}\boldsymbol{\phi}$, $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3)$. With this test function, we get by integration by part that the vertical average of the distributional form of the momentum equation (6.55) is

$$\begin{aligned}
\ulcorner \big\langle (\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon)(t), \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner &= \ulcorner \big\langle \eta_\varepsilon \varrho_{\varepsilon,0} \mathbf{u}_{\varepsilon,0}, \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner + \int_0^t \ulcorner \big\langle \eta_\varepsilon (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon), \nabla \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner \, \mathrm{d}s \\
&\quad - \int_0^t \ulcorner \big\langle \nu \, \eta_\varepsilon \nabla \mathbf{u}_\varepsilon \, , \nabla \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner \mathrm{d}s + \int_0^t \ulcorner \big\langle \varepsilon^{m-2} \eta_\varepsilon (r_\varepsilon \nabla G), \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner \, \mathrm{d}s \\
&\quad + \int_0^t \ulcorner \big\langle \mathcal{R}_\varepsilon, \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner \, \mathrm{d}s + \int_0^t \ulcorner \big\langle \eta_\varepsilon \Phi(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon), \mathcal{P}\boldsymbol{\phi} \big\rangle \urcorner \, \mathrm{d}W
\end{aligned} \tag{6.64}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and where

$$\mathcal{R}_\varepsilon = \nabla \eta_\varepsilon \cdot (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nabla \eta_\varepsilon \cdot \nu \, \mathbf{u}_\varepsilon. \tag{6.65}$$

To proceed, we make the following denotations:

$$
\begin{aligned}
\ulcorner T_\varepsilon(t)\urcorner := \ulcorner\big\langle \eta_\varepsilon\varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}, \mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner + \int_0^t \ulcorner\big\langle \eta_\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon), \nabla\mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner\,\mathrm{d}s \\
- \int_0^t \ulcorner\big\langle \nu\,\eta_\varepsilon\nabla\mathbf{u}_\varepsilon\,, \nabla\mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner\mathrm{d}s + \int_0^t \ulcorner\big\langle \varepsilon^{m-2}\eta_\varepsilon(r_\varepsilon\nabla G), \mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner\,\mathrm{d}s \\
+ \int_0^t \ulcorner\big\langle \mathcal{R}_\varepsilon, \mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner\,\mathrm{d}s \\
\ulcorner R_\varepsilon(t)\urcorner := \int_0^t \ulcorner\big\langle \eta_\varepsilon\Phi(\varrho_\varepsilon,\varrho_\varepsilon\mathbf{u}_\varepsilon), \mathcal{P}\boldsymbol{\phi}\big\rangle\urcorner\,\mathrm{d}W
\end{aligned}
$$

for all $t\in[0,T]$. Now consider any compact set $K=\overline{K}\times\mathbb{T}_1$. Then by using the continuity of the operator $\mathcal{P}$, (6.54) and the continuous embedding $W^{-1,\frac{6\gamma}{4\gamma+3}}(K)\hookrightarrow W^{-l,2}(K)$ which holds true provided $l>\frac{5}{2}$, we get that

$$
\ulcorner\mathcal{P}\,\mathrm{div}(\eta_\varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)\urcorner \in L^p\left(\Omega; L^2\left(0,T,W^{-l,2}(K)\right)\right), \quad l>\frac{5}{2}, \tag{6.66}
$$

uniformly in ε for all $p\in[1,\infty)$. Furthermore, by using (6.49) and the continuity of $\mathcal{P}$, we also get that

$$
\ulcorner\mathcal{P}\left(\nu\,\mathrm{div}(\eta_\varepsilon\nabla\mathbf{u}_\varepsilon\right)\urcorner \in L^p\left(\Omega; L^2\left(0,T,W^{-1,2}(K)\right)\right) \tag{6.67}
$$

uniformly in ε for all $p\in[1,\infty)$. Also, the continuous embedding $L^\infty(0,T;L^1(K))$ $\hookrightarrow L^2(0,T;W^{-l,2}(K))$ gives

$$
\ulcorner\mathcal{P}\left(\eta_\varepsilon r_\varepsilon\nabla G\right)\urcorner \in L^1\left(\Omega; L^2\left(0,T,W^{-l,2}(K)\right)\right). \tag{6.68}
$$

Indeed, by using the aforementioned embedding, the continuity of $\mathcal{P}$, (6.17), Hölder's inequality and (6.51), we have that

$$
\begin{aligned}
\mathbb{E}\|\ulcorner\mathcal{P}(\eta_\varepsilon r_\varepsilon\nabla G)\urcorner\|_{L^2\left(0,T,W^{-l,2}(K)\right)} &\leq c\mathbb{E}\bigg(\|\nabla G\|_{L^\infty_x}\sup_{t\in(0,T)}\|\ulcorner r_\varepsilon\urcorner\|_{L^1(K)}\bigg) \\
&\leq c\,\mathbb{E}\sup_{t\in(0,T)}\|r_\varepsilon\|_{L^{\min\{2,\gamma\}}(K)}\leq c
\end{aligned} \tag{6.69}
$$

uniformly in ε. The residual term (6.65) is comparable to (6.66), (6.67) and is in fact, of lower order. Subsequently, by combining (6.66), (6.67) and (6.68), it follows that

$$\partial_t \ulcorner T_\varepsilon(t) \urcorner \in L^1\left(\Omega; L^2\left(0, T, W^{-l,2}(K)\right)\right) \tag{6.70}$$

uniformly in ε since $\varepsilon^{m-2} \ll 1$ for $m > 3$. It follows from (6.70) that the mean

$$\mathbb{E}\, \| \ulcorner T_\varepsilon \urcorner \|_{C^\vartheta\left([0,T]; W^{-l,2}(K)\right)} \leq c \tag{6.71}$$

is bounded uniformly in ε for $\vartheta \in [0, \frac{1}{2}]$. Again, much like the proof of [11, Proposition 3.6], one gets by using (6.6) and (6.9) that

$$\begin{aligned}
&\mathbb{E}\, \| \ulcorner R_\varepsilon(t) \urcorner - \ulcorner R_\varepsilon(s) \urcorner \|^\theta_{W^{-l,2}(K)} \leq c\, \mathbb{E} \left\| \int_s^t \Phi\left(\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \mathrm{d}W \right\|^\theta_{W^{-l,2}(K)} \\
&\leq c\, \mathbb{E} \left(\int_s^t \sum_{k \in \mathbb{N}} \| \mathbf{g}_k\left(x, \varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \|^2_{W^{-l,2}(K)} \, \mathrm{d}\tau \right)^{\frac{\theta}{2}} \\
&\leq c\, \mathbb{E} \left(\int_s^t \sum_{k \in \mathbb{N}} \| \mathbf{g}_k\left(x, \varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon\right) \|^2_{L^1(K)} \, \mathrm{d}\tau \right)^{\frac{\theta}{2}} \\
&\leq c\, \mathbb{E} \left(\int_s^t \int_K (1 + \varrho_\varepsilon^\gamma + \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2) \mathrm{d}x \, \mathrm{d}\tau \right)^{\frac{\theta}{2}} \\
&\leq c\, |t-s|^{\frac{\theta}{2}} \left(1 + \mathbb{E} \sup_{t \in [0,T]} \| \varrho_\varepsilon \|^{\theta\gamma/2}_{L^\gamma(K)} + \mathbb{E} \sup_{t \in [0,T]} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \|^\theta_{L^2(K)} \right)^{\frac{\theta}{2}} \\
&\leq c\, |t-s|^{\frac{\theta}{2}}.
\end{aligned}$$

In the last estimate above, we have used (6.50) and (6.52). We now apply Kolmogorov's continuity criterion and then combining with (6.71), we get that

$$\mathbb{E}\, \| \ulcorner \mathcal{P}\left(\eta_\varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon\right)(t) \urcorner \|_{C^\vartheta\left([0,T]; W^{-l,2}(K)\right)} \leq c. \tag{6.72}$$

Finally, we use the compact embedding (see [87, Corollary B.2])

$$L^\infty\left(0, T; L^{\frac{2\gamma}{\gamma+1}}(K)\right) \cap C^\vartheta\left([0,T]; W^{-l,2}(K)\right) \hookrightarrow C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}(K)\right) \tag{6.73}$$

and (6.53) to finish the proof. □

Lemma 6.3.11. The collection $\{\nu^{\varepsilon,\delta}\, ;\, \varepsilon, \delta \in (0,1)\}$ is tight on χ^J.

Proof. This is similar to Proposition 3.3.16 or [11, Corollary 3.7]. □

Now by the Jakubowski–Shorokhod representation theorem, Theorem 2.4.29, we gain the following result.

Proposition 6.3.12. For any subsequence $\{\nu^{\varepsilon_n,\delta_n}\,;\, n\in\mathbb{N}\}$, there exists a further subsequence (not relabelled), a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ with χ^J-valued random variables

$$(\hat{\varrho},\hat{\mathbf{U}},\hat{\mathbf{m}},\check{\varrho},\check{\mathbf{U}},\check{\mathbf{m}},\tilde{W})\text{ and }(\hat{\varrho}_{\varepsilon_n},\hat{\mathbf{u}}_{\varepsilon_n},\hat{\mathbf{m}}_{\varepsilon_n},\check{\varrho}_{\delta_n},\check{\mathbf{u}}_{\delta_n},\check{\mathbf{m}}_{\delta_n},\tilde{W}_n),\quad n\in\mathbb{N}$$

and $\varepsilon_n,\delta_n\in(0,1)$ such that as $n\to\infty$, we have

1. $\tilde{\mathbb{P}}((\hat{\varrho}_{\varepsilon_n},\hat{\mathbf{u}}_{\varepsilon_n},\hat{\mathbf{m}}_{\varepsilon_n},\check{\varrho}_{\delta_n},\check{\mathbf{u}}_{\delta_n},\check{\mathbf{m}}_{\delta_n},\tilde{W}_n)\in\cdot)=\nu^{\varepsilon_n,\delta_n}(\cdot)$,
2. $\tilde{\mathbb{P}}((\hat{\varrho},\hat{\mathbf{U}},\hat{\mathbf{m}},\check{\varrho},\check{\mathbf{U}},\check{\mathbf{m}},\tilde{W})\in\cdot)=\nu(\cdot)$ is a Radon measure,
3. the sequences $(\hat{\varrho}_{\varepsilon_n},\hat{\mathbf{u}}_{\varepsilon_n},\hat{\mathbf{m}}_{\varepsilon_n},\check{\varrho}_{\delta_n},\check{\mathbf{u}}_{\delta_n},\check{\mathbf{m}}_{\delta_n},\tilde{W}_n)$ converges $\tilde{\mathbb{P}}$-a.s. to $(\hat{\varrho},\hat{\mathbf{U}},\hat{\mathbf{m}},\check{\varrho},\check{\mathbf{U}},\check{\mathbf{m}},\tilde{W})$ in the topology of χ^J.

In particular, the joint law of $(\hat{\varrho}_{\varepsilon_n},\hat{\mathbf{u}}_{\varepsilon_n},\hat{\mathbf{m}}_{\varepsilon_n},\check{\varrho}_{\delta_n},\check{\mathbf{u}}_{\delta_n},\check{\mathbf{m}}_{\delta_n})$, i.e. $\mu^{\varepsilon_n,\delta_n}$, converges weakly to the measure $\mu(\cdot)=\tilde{\mathbb{P}}((\hat{\varrho},\hat{\mathbf{U}},\hat{\mathbf{m}},\check{\varrho},\check{\mathbf{U}},\check{\mathbf{m}})\in\cdot)$.

To extend this new probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ into a stochastic basis, we endow it with a filtration. To do this, let us first define a restriction operator $\mathbf{r}_t$ by

$$\mathbf{r}_t: X\to X|_{[0,t]},\quad f\mapsto f|_{[0,t]}, \tag{6.74}$$

for $t\in[0,T]$ and $X\in\{\chi_\varrho,\chi_{\mathbf{u}},\chi_W\}$. We observe that $\mathbf{r}_t$ is a continuous map. We can therefore construct $\tilde{\mathbb{P}}$-augmented canonical filtrations for

$$(\hat{\varrho}_{\varepsilon_n},\hat{\mathbf{u}}_{\varepsilon_n},\check{\varrho}_{\delta_n},\check{\mathbf{u}}_{\delta_n},\tilde{W}_n)\text{ and }(\hat{\varrho},\hat{\mathbf{U}},\check{\varrho},\check{\mathbf{U}},\tilde{W})$$

respectively by setting

$$\tilde{\mathscr{F}}_t^n = \sigma\left(\sigma(\mathbf{r}_t\hat{\varrho}_{\varepsilon_n}, \mathbf{r}_t\hat{\mathbf{u}}_{\varepsilon_n}, \mathbf{r}_t\check{\varrho}_{\delta_n}, \mathbf{r}_t\check{\mathbf{u}}_{\delta_n}, \mathbf{r}_t\tilde{W}_n) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\right),$$
$$\tilde{\mathscr{F}}_t = \sigma\left(\sigma(\mathbf{r}_t\hat{\varrho}, \mathbf{r}_t\hat{\mathbf{U}}, \mathbf{r}_t\check{\varrho}, \mathbf{r}_t\check{\mathbf{U}}, \mathbf{r}_t\tilde{W}) \cup \{N \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(N) = 0\}\right),$$

for $t \in [0, T]$.

6.3.13 Identification of the limit

Having established the limits of the family of sequences in Proposition 6.3.12, we now identify them with weak martingale solutions of (6.4). In fact, as a consequence of the 2-D uniqueness theorem given in Theorem 6.2.12, we show that the corresponding random variables coincides. We state this in the theorem below.

Theorem 6.3.14. *The pair* $[(\tilde{\Omega}, \tilde{\mathscr{F}}_t, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \hat{\mathbf{U}}, \tilde{W}]$ *and* $[(\tilde{\Omega}, \tilde{\mathscr{F}}_t, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \check{\mathbf{U}}, \tilde{W}]$ *are each a weak martingale solution of* (6.4) *in the sense of Definition 6.2.10 defined on the same stochastic basis. Moreover*

$$\hat{\varrho} = \check{\varrho} = 1, \quad \hat{\mathbf{m}} = \hat{\mathbf{U}}_h, \quad \check{\mathbf{m}} = \check{\mathbf{U}}_h \tag{6.75}$$

$\tilde{\mathbb{P}}$*-a.s. where* $\hat{\mathbf{U}}_h = \hat{\mathbf{U}}_h(x_h)$ *satisfies* $\hat{\mathbf{U}} = [\hat{\mathbf{U}}_h(x_h), 0]$ *and similarly for* $\check{\mathbf{U}}$*.*

The proof of Theorem 6.3.14 follows several steps. First of all, we show that on the new probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}_t, \tilde{\mathbb{P}})$, any pair of approximate subsequence of functions $(\hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n})$ and $(\check{\varrho}_{\delta_n}, \check{\mathbf{u}}_{\delta_n})$ also solves the system (6.1).

Lemma 6.3.15. The following subsequence which are defined on the same stochastic basis

$$[(\tilde{\Omega}, \tilde{\mathscr{F}}_t, (\tilde{\mathscr{F}}_t^n), \tilde{\mathbb{P}}), \hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n}, \tilde{W}_n] \text{ and } [(\tilde{\Omega}, \tilde{\mathscr{F}}_t^n, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \check{\varrho}_{\delta_n}, \check{\mathbf{u}}_{\delta_n}, \tilde{W}_n]$$

are each a finite energy weak martingale solutions of (6.1) with initial law Λ_n

Proof. This follows from the equality of laws from Proposition 6.3.12 and Theorem

2.4.31. □

Until otherwise stated, we now concentrate our attention on the analysis of the sequence $\left(\hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n}, \tilde{W}_n\right)$, which as stated earlier, shares the same stochastic basis with the sequence $\left(\check{\varrho}_{\delta_n}, \check{\mathbf{u}}_{\delta_n}, \tilde{W}_n\right)$. Analysis of the later is mere repetition.

6.4 Analysis of the momentum sequence

In the previous section, we have obtained two limits of a pair of sequences that have been shown to be weak martingale solutions to (6.4) in the sense of Definition 6.2.10. In this section, we wish to give a rigorous analysis into the momentum function by studying its various components.

6.4.1 Acoustic equation and its Strichartz estimates

In this section, we establish dispersive estimates for the acoustic wave equation obtained by projecting the vector quantities in the momentum balance equation unto gradient vector fields via Helmholtz decomposition. Our main result in this section is the proof of the following lemma.

Lemma 6.4.2. Let $\Delta_{\mathcal{O}}^{-1}$ represent the inverse of the Laplace operator on $\mathcal{O} = \mathbb{R}^2 \times \mathbb{T}_1$, let η_ε be as defined in (6.62) and set $\mathcal{Q} = \nabla \Delta_{\mathcal{O}}^{-1} \mathrm{div}$. Then there exist a subsequence (not relabelled) such that the following $\tilde{\mathbb{P}}$-a.s convergence holds

$$\mathcal{Q}(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}) \to 0 \quad \text{in} \quad L^2(0,T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(\mathcal{O})), \tag{6.76}$$

as $n \to \infty$.

Proof. Here, we follow the approach of [36, Section 3.2.1] applied to the mild form of the mass and momentum balance equation.

Given Proposition 6.3.12 and Lemma 6.3.15, we know $(\hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n})$ to be a weak solution of (6.55). By multiplying the continuity equation $(6.55)_1$ with the cut-off function

η_ε introduced in (6.62), we expect $(\hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n})$ to be a distributional solution of

$$\varepsilon^m \,\mathrm{d}\big[\eta_\varepsilon \hat{r}_{\varepsilon_n}\big] + \mathrm{div}\big[\eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]\mathrm{d}t = \nabla\eta_\varepsilon \cdot \big(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\big)\,\mathrm{d}t. \tag{6.77}$$

However by using (6.62), in particular $|\nabla\eta_\varepsilon(x_h)| \leq 2\varepsilon^\alpha$, we can conclude from (6.53) and Proposition 6.3.12 that

$$\nabla\eta_\varepsilon \cdot \big(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\big) = \varepsilon^\alpha\, \hat{F}_{\varepsilon_n} \tag{6.78}$$

for some $\hat{F}_{\varepsilon_n}$ belonging to

$$\hat{F}_{\varepsilon_n} \in L^p\big(\tilde{\Omega}; L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}\mathcal{O})\big) \tag{6.79}$$

uniformly in n. We can therefore rewrite (6.77) as

$$\begin{aligned}\varepsilon^m\big\langle \eta_\varepsilon\hat{r}_{\varepsilon_n}(t)\,,\,\varphi\big\rangle - \int_0^t \big\langle \eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\,,\,\nabla\varphi\big\rangle \mathrm{d}s &= \varepsilon^m\big\langle \eta_\varepsilon\hat{r}_{\varepsilon_n}(0)\,,\,\varphi\big\rangle \\ &+ \varepsilon^\alpha \int_0^t \big\langle \hat{F}_{\varepsilon_n}\,,\,\varphi\big\rangle \mathrm{d}s\end{aligned} \tag{6.80}$$

for all $t \in [0,T]$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$. Similar to (6.77), the corresponding momentum balance equation $(6.55)_2$ becomes

$$\begin{aligned}&\varepsilon^m\,\mathrm{d}\big[\eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big] + \gamma\nabla\big[\eta_\varepsilon\hat{r}_{\varepsilon_n}\big]\mathrm{d}t = \varepsilon^m\,\mathrm{div}\big[\eta_\varepsilon\mathbb{S}(\nabla\hat{\mathbf{u}}_{\varepsilon_n})\big]\mathrm{d}t \\ &- \varepsilon^m\,\mathrm{div}\big[\eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n})\big]\mathrm{d}t - \varepsilon^{m-1}\eta_\varepsilon\big(\mathbf{e}_3\times\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\big)\,\mathrm{d}t \\ &- \frac{1}{\varepsilon^m}\nabla\big[\eta_\varepsilon\big(\hat{\varrho}^\gamma_{\varepsilon_n} - \gamma(\hat{\varrho}_{\varepsilon_n} - \overline{\varrho}_\varepsilon) - \overline{\varrho}^\gamma_\varepsilon\big)\big]\mathrm{d}t \\ &+ \varepsilon^{2(m-1)}\eta_\varepsilon(\hat{r}_{\varepsilon_n}\nabla G)\,\mathrm{d}t + \varepsilon^m\,\eta_\varepsilon\,\Phi(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\mathrm{d}\tilde{W}_{\varepsilon_n} \\ &- \varepsilon^m\,\nabla\eta_\varepsilon\cdot\big[\mathbb{S}(\nabla\hat{\mathbf{u}}_{\varepsilon_n})\big]\mathrm{d}t + \varepsilon^m\,\nabla\eta_\varepsilon\cdot(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n})\mathrm{d}t \\ &+ \gamma\nabla\eta_\varepsilon\,\hat{r}_{\varepsilon_n}\mathrm{d}t + \nabla\eta_\varepsilon\,\frac{1}{\varepsilon^m}\big[\hat{\varrho}^\gamma_{\varepsilon_n} - \gamma(\hat{\varrho}_{\varepsilon_n} - \overline{\varrho}_\varepsilon) - \overline{\varrho}^\gamma_\varepsilon\big]\mathrm{d}t \\ &=: \sum_{j=1}^{10} I_j\end{aligned} \tag{6.81}$$

which is to be understood in the distributional sense, i.e. for $t \in [0,T]$, the equality

$$
\begin{aligned}
&\varepsilon^m\big\langle\eta_\varepsilon(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})(t)\,,\,\boldsymbol{\varphi}\big\rangle-\gamma\int_0^t\big\langle\eta_\varepsilon\hat r_{\varepsilon_n}\,,\,\mathrm{div}\boldsymbol{\varphi}\big\rangle\,\mathrm{d}s=\varepsilon^m\big\langle\eta_\varepsilon(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})(0)\,,\,\boldsymbol{\varphi}\big\rangle\\
&-\varepsilon^m\int_0^t\big\langle\eta_\varepsilon\mathbb{S}(\nabla\hat{\mathbf{u}}_{\varepsilon_n})\,,\,\nabla\boldsymbol{\varphi}\big\rangle\mathrm{d}s+\varepsilon^m\int_0^t\big\langle\eta_\varepsilon(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n})\,,\,\nabla\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&-\varepsilon^{m-1}\int_0^t\big\langle\eta_\varepsilon\big(\mathbf{e}_3\times\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\big)\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&+\frac{1}{\varepsilon^m}\int_0^t\big\langle\eta_\varepsilon\big(\hat\varrho^\gamma_{\varepsilon_n}-\gamma(\hat\varrho_{\varepsilon_n}-\overline{\varrho}_\varepsilon)-\overline{\varrho}^\gamma_\varepsilon\big)\,,\,\mathrm{div}\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&+\varepsilon^{2(m-1)}\int_0^t\big\langle\eta_\varepsilon(\hat r_{\varepsilon_n}\nabla G)\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s+\varepsilon^m\int_0^t\big\langle\eta_\varepsilon\,\Phi(\hat\varrho_{\varepsilon_n},\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\mathrm{d}\tilde W_{\varepsilon_n}\,,\,\boldsymbol{\varphi}\big\rangle\\
&-\varepsilon^m\int_0^t\big\langle\nabla\eta_\varepsilon\cdot\mathbb{S}(\nabla\hat{\mathbf{u}}_{\varepsilon_n})\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s+\varepsilon^m\int_0^t\big\langle\nabla\eta_\varepsilon\cdot(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n})\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&+\gamma\int_0^t\big\langle\nabla\eta_\varepsilon\,\hat r_{\varepsilon_n}\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s+\frac{1}{\varepsilon^m}\int_0^t\big\langle\nabla\eta_\varepsilon\big[\hat\varrho^\gamma_{\varepsilon_n}-\gamma(\hat\varrho_{\varepsilon_n}-\overline{\varrho}_\varepsilon)-\overline{\varrho}^\gamma_\varepsilon\big]\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&=:\sum_{j=1}^{10}I_j
\end{aligned}
\tag{6.82}
$$

holds for any $\boldsymbol{\varphi}\in C_c^\infty(\mathbb{R}^3)$. Regularity for the terms in (6.82) follows from the uniform estimates shown in Section 6.3.1 and the equality of law given by Proposition 6.3.12. The terms $I_7,\dots,I_{10}$ in (6.82) are even of lower order. Again by using (6.62), we can do a similar analysis as in (6.78) for the momentum equation (6.82) to get

$$
\begin{aligned}
\varepsilon^m\big\langle\eta_\varepsilon(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})(t)\,,\,\boldsymbol{\varphi}\big\rangle-\gamma\int_0^t\big\langle\eta_\varepsilon\hat r_{\varepsilon_n}\,,\,\mathrm{div}\boldsymbol{\varphi}\big\rangle\mathrm{d}s&=\varepsilon^m\big\langle\eta_\varepsilon(\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})(0)\,,\,\boldsymbol{\varphi}\big\rangle\\
&-\big(\varepsilon^m+\varepsilon^{2(m-1-\alpha)}\big)\int_0^t\big\langle\hat{\mathbb{F}}_{\varepsilon_n}\,,\,\nabla\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&+\int_0^t\big\langle\big(\varepsilon^{m-1}+\varepsilon^\alpha+\varepsilon^{2(m-1-\alpha)}\big)\,\hat{\mathbf{F}}_{\varepsilon_n}\,,\,\boldsymbol{\varphi}\big\rangle\mathrm{d}s\\
&+\varepsilon^m\int_0^t\big\langle\eta_\varepsilon\Phi(\hat\varrho_{\varepsilon_n},\hat\varrho_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\mathrm{d}\tilde W_{\varepsilon_n}\,,\,\boldsymbol{\varphi}\big\rangle
\end{aligned}
\tag{6.83}
$$

for any $t\in[0,T]$ and any $\boldsymbol{\varphi}\in C_c^\infty(\mathbb{R}^3)$ and where

$$
\begin{aligned}
&\hat{\mathbb{F}}_{\varepsilon_n}\in L^p\big(\tilde\Omega;L^2(0,T;L^1_{\mathrm{loc}}(\mathcal{O}))\big),\\
&\hat{\mathbf{F}}_{\varepsilon_n}\in L^p\big(\tilde\Omega;L^2(0,T;L^1_{\mathrm{loc}}(\mathcal{O}))\big)
\end{aligned}
\tag{6.84}
$$

uniformly in ε_n. We can mollify (6.80) and (6.83) by convolution with the usual mollifier $\wp_\kappa$ to get for a.e. $(\omega, t, x) \in \tilde{\Omega} \times [0,T] \times \mathcal{O}$,

$$\varepsilon^m \,\mathrm{d}\big[\eta_\varepsilon \hat{r}_{\varepsilon_n}\big]_\kappa + \mathrm{div}\big[\eta_\varepsilon (\hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa \mathrm{d}t = \varepsilon^\alpha \hat{F}_{\varepsilon_n,\kappa}\,\mathrm{d}t \tag{6.85}$$

and

$$\begin{aligned} \varepsilon^m \,\mathrm{d}\big[\eta_\varepsilon (\hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa + \gamma \nabla \big[\eta_\varepsilon \hat{r}_{\varepsilon_n}\big]_\kappa \mathrm{d}t = \big[A_\varepsilon(m,\alpha)\,\mathrm{div}\,\mathbb{F}_{\varepsilon_n,\kappa} \\ + B_\varepsilon(m,\alpha)\,\mathbf{F}_{\varepsilon_n,\kappa}\big]\mathrm{d}t + \varepsilon^m \,\big[\eta_\varepsilon \Phi(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa \mathrm{d}\tilde{W}_{\varepsilon_n}. \end{aligned} \tag{6.86}$$

respectively with

$$A_\varepsilon(m,\alpha) = \varepsilon^m + \varepsilon^{2(m-1-\alpha)}, \quad B_\varepsilon(m,\alpha) = \varepsilon^{m-1} + \varepsilon^\alpha + \varepsilon^{2(m-1-\alpha)}. \tag{6.87}$$

Let $\mathcal{Q} = \nabla \Delta_{\mathcal{O}}^{-1}\mathrm{div}$ and $\mathcal{P}$ be respectively, the gradient and solenoidal parts according to Helmholtz decomposition with the identity operator satisfying $\mathrm{Id} = \mathcal{Q} + \mathcal{P}$. Now define the function $\hat{\Psi}_{\varepsilon_n,\kappa} = \Delta_{\mathcal{O}}^{-1}\mathrm{div}\big[\eta_\varepsilon (\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa$ so that the relation $\nabla \hat{\Psi}_{\varepsilon_n,\kappa} = \mathcal{Q}\big[\eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa$ holds. We can then recast (6.85) as

$$\varepsilon^m \,\mathrm{d}\hat{\varphi}_{\varepsilon_n,\kappa} + \Delta \hat{\Psi}_{\varepsilon_n,\kappa}\,\mathrm{d}t = \varepsilon^\alpha \hat{F}_{\varepsilon_n,\kappa}\,\mathrm{d}t \tag{6.88}$$

where

$$\hat{\varphi}_{\varepsilon_n,\kappa} := \big[\eta_\varepsilon \hat{r}_{\varepsilon_n}\big]_\kappa = \Big[\eta_\varepsilon \frac{\hat{\varrho}_{\varepsilon_n} - \overline{\varrho}_\varepsilon}{\varepsilon^m}\Big]_\kappa.$$

Applying the operator $\mathcal{Q}$ to (6.86) also yields

$$\begin{aligned} \varepsilon^m \,\mathrm{d}\big[\nabla \hat{\Psi}_{\varepsilon_n,\kappa}\big] + \gamma \nabla \hat{\varphi}_{\varepsilon_n,\kappa}\mathrm{d}t = \big[A_\varepsilon(m,\alpha)\mathcal{Q}\mathrm{div}\,\mathbb{F}_{\varepsilon_n,\kappa} \\ + B_\varepsilon(m,\alpha)\mathcal{Q}\,\mathbf{F}_{\varepsilon_n,\kappa}\big]\mathrm{d}t + \varepsilon^m \,\mathcal{Q}\big[\eta_\varepsilon \Phi(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa \mathrm{d}\tilde{W}_{\varepsilon_n}. \end{aligned} \tag{6.89}$$

Equations (6.88) and (6.89) is equivalent to the system

$$\begin{aligned}\varepsilon^m \mathrm{d}\begin{bmatrix}\hat{\varphi}_{\varepsilon_n,\kappa}\\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa}\end{bmatrix} &= \mathcal{A}\begin{bmatrix}\hat{\varphi}_{\varepsilon_n,\kappa}\\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa}\end{bmatrix}\mathrm{d}t + \varepsilon^m\begin{bmatrix}\frac{\varepsilon^\alpha}{\varepsilon^m}\hat{F}_{\varepsilon_n,\kappa}\\ 0\end{bmatrix}\mathrm{d}t\\ &+ \varepsilon^m\begin{bmatrix}0\\ \frac{B_\varepsilon}{\varepsilon^m}\mathcal{Q}\hat{\mathbf{F}}_{\varepsilon_n,\kappa}\end{bmatrix}\mathrm{d}t + \varepsilon^m\begin{bmatrix}0\\ \frac{A_\varepsilon}{\varepsilon^m}\mathcal{Q}\mathrm{div}\,\hat{\mathbb{F}}_{\varepsilon_n,\kappa}\end{bmatrix}\mathrm{d}t + \varepsilon^m\begin{bmatrix}0\\ \mathcal{Q}\eta_\varepsilon\hat{\Phi}_{\varepsilon_n,\kappa}\end{bmatrix}\mathrm{d}W_\varepsilon.\end{aligned} \tag{6.90}$$

Here $\hat{\Phi}_{\varepsilon_n,\kappa} := \Phi\left(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\right)_\kappa$ and the operator $\mathcal{A}$ is given by

$$\mathcal{A} := -\begin{bmatrix}0 & \mathrm{div}\\ \gamma\nabla & 0\end{bmatrix}. \tag{6.91}$$

Now let $E = L^2(\mathcal{O})\times L^2(\mathcal{O};\mathbb{R}^N)$ and consider the operator given by $S(t) = e^{t\mathcal{A}}$. We observe that $(S(t))_{t\geq 0}$, as a function of t, is a strongly continuous semigroup since,

$$S(0) = \mathbb{1}, \quad S(t+s) = S(t)S(s), \quad \lim_{t\downarrow 0} S(t)\mathbf{f} = \mathbf{f}$$

for all $\mathbf{f} = [\varphi, \nabla\Psi]^T \in E$. Moreover $\mathcal{A}$ is linear and

$$\mathcal{A}\mathbf{f} = \lim_{t\downarrow 0}\frac{S(t)\mathbf{f}-\mathbf{f}}{t} = \left.\frac{\mathrm{d}S(t)\mathbf{f}}{\mathrm{d}t}\right|_{t=0} = e^{t\mathcal{A}}\mathcal{A}\mathbf{f}\mid_{t=0}.$$

Hence, $\mathcal{A}$ is an infinitesimal generator of the strongly continuous semigroup $S(t)$ with domain

$$\begin{aligned}\mathrm{Dom}(\mathcal{A}) &= \left\{\mathbf{f}\in E \,:\, \lim_{t\downarrow 0}\frac{S(t)\mathbf{f}-\mathbf{f}}{t} \text{ exists}\right\}\\ &= \left\{\mathbf{f} = [\varphi,\nabla\Psi]^T \,:\, \varphi\in W^{1,2}(\mathcal{O}),\ \nabla\Psi\in L^2(\mathcal{O}),\ \mathrm{div}\,\nabla\Psi\in L^2(\mathcal{O})\right\}\end{aligned}$$

Proposition 6.4.3. Assume that $\mathcal{A}\,:\,\mathrm{Dom}(\mathcal{A})\subset E\longrightarrow E$ is an infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on E and that

$$\hat{\Phi}_{\varepsilon_n,\kappa}\in\mathcal{N}^2_W\big(0,T;L_2(\mathfrak{U};W^{-l,2}(\mathcal{O}))\big), \quad l>5/2,$$

recall (4.5). Then a weak solution of (6.90) is also a mild solution.

Proof. Compare with Proposition 4.7.7. □

As a result of Proposition 6.4.3, we can rewrite Eq. (6.90), after rescaling in time, in the mild form[2]

$$\begin{aligned}
&\begin{bmatrix} \hat{\varphi}_{\varepsilon_n,\kappa} \\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa} \end{bmatrix}(t) = S\left(\frac{t}{\varepsilon^m}\right)\begin{bmatrix} \hat{\varphi}_{\varepsilon_n,\kappa}(0) \\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa}(0) \end{bmatrix} + \int_0^t S\left(\frac{t-s}{\varepsilon^m}\right)\begin{bmatrix} \frac{\varepsilon^\alpha}{\varepsilon^m}\hat{F}_{\varepsilon_n,\kappa} \\ 0 \end{bmatrix}\mathrm{d}s \\
&+\int_0^t S\left(\frac{t-s}{\varepsilon^m}\right)\begin{bmatrix} 0 \\ \frac{B_\varepsilon}{\varepsilon^m}\mathcal{Q}\hat{\mathbf{F}}_{\varepsilon_n,\kappa} \end{bmatrix}\mathrm{d}s + \int_0^t S\left(\frac{t-s}{\varepsilon^m}\right)\begin{bmatrix} 0 \\ \frac{A_\varepsilon}{\varepsilon^m}\mathcal{Q}\mathrm{div}\,\hat{\mathbb{F}}_{\varepsilon_n,\kappa} \end{bmatrix}\mathrm{d}s \\
&+\int_0^t S\left(\frac{t-s}{\varepsilon^m}\right)\begin{bmatrix} 0 \\ \mathcal{Q}\eta_\varepsilon\hat{\Phi}_{\varepsilon_n,\kappa} \end{bmatrix}\mathrm{d}\tilde{W}_{s,n} \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{6.92}$$

where

$$S(t)\begin{bmatrix} \hat{\varphi}_{0,\varepsilon_n,\kappa}(\cdot) \\ \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa}(\cdot) \end{bmatrix} = \begin{bmatrix} \hat{\varphi}_{\varepsilon_n,\kappa}(\cdot,t) \\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa}(\cdot,t) \end{bmatrix} \tag{6.93}$$

is the solution to the homogeneous PDE

$$\begin{aligned}
\mathrm{d}\hat{\varphi}_{\varepsilon_n,\kappa} + \Delta\hat{\Psi}_{\varepsilon_n,\kappa}\,\mathrm{d}t &= 0, \\
\mathrm{d}\nabla\hat{\Psi}_{\varepsilon_n,\kappa} + \gamma\nabla\hat{\varphi}_{\varepsilon_n,\kappa}\,\mathrm{d}t &= 0, \\
\hat{\varphi}_{\varepsilon_n,\kappa}(0) = \hat{\varphi}_{0,\varepsilon_n,\kappa}; \quad \nabla\hat{\Psi}_{\varepsilon_n,\kappa}(0) &= \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa}.
\end{aligned} \tag{6.94}$$

C.f. the purely deterministic case [42, Eq. 8.111]. Using Fourier transforms (in space), we obtain for a.e. $(\omega, x) \in \Omega \times \mathcal{O}$, an exact solution to (6.94) given by the

[2]This mild formulation is essentially a stochastic version of the Duhamel's formula with the added stochastic convolution term given by the noise.

pair

$$
\begin{aligned}
\nabla\hat{\Psi}_{\varepsilon_n,\kappa}(\cdot,t) &= \frac{1}{2}\exp\left(i\sqrt{-\gamma\Delta}t\right)\left(\nabla\hat{\Psi}_{0,\varepsilon_n,\kappa} + \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\hat{\varphi}_{0,\varepsilon_n,\kappa}\right) \\
&\quad + \frac{1}{2}\exp\left(-i\sqrt{-\gamma\Delta}t\right)\left(\nabla\hat{\Psi}_{0,\varepsilon_n,\kappa} - \frac{i\sqrt{\gamma}}{\sqrt{-\Delta}}\nabla\hat{\varphi}_{0,\varepsilon_n,\kappa}\right), \\
\hat{\varphi}_{\varepsilon_n,\kappa}(\cdot,t) &= \frac{1}{2}\exp\left(i\sqrt{-\gamma\Delta}t\right)\left(\hat{\varphi}_{0,\varepsilon_n,\kappa} - \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\hat{\Psi}_{0,\varepsilon_n,\kappa}\right) \\
&\quad + \frac{1}{2}\exp\left(-i\sqrt{-\gamma\Delta}t\right)\left(\hat{\varphi}_{0,\varepsilon_n,\kappa} + \frac{i\sqrt{-\Delta}}{\sqrt{\gamma}}\hat{\Psi}_{0,\varepsilon_n,\kappa}\right).
\end{aligned}
\tag{6.95}
$$

Remark 6.4.4. Note that by substitution, the problem (6.94) (and thus its solution (6.95)), may be recast as a single system of PDEs. A similar remark holds for the inhomogeneous counterpart of (6.94).

We now state a lemma, the proof of which follows by taking expectation in [36, Lemma 3.1].

Lemma 6.4.5. Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Then the inequality

$$
\mathbb{E}\left\|\phi(x_h)\,\exp(i\sqrt{-\gamma\Delta}t)\,[\mathbf{f}]\right\|^2_{L^2(\mathbb{R}\times\mathcal{O})} \le c(\phi)\mathbb{E}\,\|\,\mathbf{f}\,\|^2_{L^2(\mathcal{O})}
$$

holds for $\mathbf{f} \in L^2(\Omega\times\mathcal{O})$.

With Lemma 6.4.5 in hand, we are able to estimate the right-hand of (6.92). To see this, we first notice that by rescaling (6.93) in time, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|S\left(\frac{t}{\varepsilon^m}\right)\begin{bmatrix}\hat{\varphi}_{0,\varepsilon_n,\kappa}\\ \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa}\end{bmatrix}\right\|^2_{L^2((0,T)\times\mathcal{O})} \le \mathbb{E}\left\|S\left(\frac{t}{\varepsilon^m}\right)\begin{bmatrix}\hat{\varphi}_{0,\varepsilon_n,\kappa}\\ \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa}\end{bmatrix}\right\|^2_{L^2(\mathbb{R}\times\mathcal{O})} \\
&\le \varepsilon^m\,\mathbb{E}\left\|S\left(t\right)\begin{bmatrix}\hat{\varphi}_{0,\varepsilon_n,\kappa}\\ \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa}\end{bmatrix}\right\|^2_{L^2(\mathbb{R}\times\mathcal{O})} = \varepsilon^m\,\mathbb{E}\left\|\begin{bmatrix}\hat{\varphi}_{\varepsilon_n,\kappa}(t)\\ \nabla\hat{\Psi}_{\varepsilon_n,\kappa}(t)\end{bmatrix}\right\|^2_{L^2(\mathbb{R}\times\mathcal{O})}.
\end{aligned}
\tag{6.96}
$$

However since the final term in (6.96) above has entries satisfying (6.95), we can use Lemma 6.4.5 to show that these terms are controlled. In particular, for $g_{0,\varepsilon_n,\kappa}$ denoting the initial data on the right-hand side of (6.95), we gain by using Lemma

6.4.5 that the estimate

$$\begin{aligned}
\mathbb{E}\left\| \exp\left(i\sqrt{-\gamma\Delta}t\right) g_{0,\varepsilon_n,\kappa} \right\|^2_{L^2(\mathbb{R}\times\mathcal{O})} &\lesssim \mathbb{E}\,\|\, g_{0,\varepsilon_n,\kappa}\|^2_{L^2(\mathcal{O})} \\
&\lesssim \|\,\wp_\kappa\|^2_{L^p(\mathcal{O})}\,\mathbb{E}\,\|\, g_{0,\varepsilon_n}\|^2_{L^q(\mathcal{O})} \\
&\lesssim_\kappa \mathbb{E}\,\|\, g_{0,\varepsilon_n}\|^2_{L^q(\mathcal{O})}
\end{aligned} \tag{6.97}$$

holds for $1/p + 1/q = 3/2$. Subsequently we can use the boundedness of the initial law (6.32) and the mollifier $\wp_\kappa$ to conclude that

$$\mathbb{E}\left\| S\left(\frac{t}{\varepsilon^m}\right) \begin{bmatrix} \hat{\varphi}_{0,\varepsilon_n,\kappa} \\ \nabla\hat{\Psi}_{0,\varepsilon_n,\kappa} \end{bmatrix} \right\|^2_{L^2((0,T)\times\mathcal{O})} \leq c(\kappa)\,\varepsilon^m. \tag{6.98}$$

Uniform estimates for the terms $J_2, \ldots, J_4$ in (6.92) follows a similar argument as in (4.87)–(4.90) (c.f. [36, Eq. 3.21]). Indeed with the uniform estimates Lemma 6.3.5, (6.56), (6.58) as well as the equality of laws given in Proposition 6.3.12 in hand, we have that,

$$\begin{aligned}
&\mathbb{E}\left\| \int_0^t S\left(\frac{t-s}{\varepsilon^m}\right) D_\varepsilon\,\mathbf{f}_{\varepsilon_n,\kappa}\,\mathrm{d}s \right\|^2_{L^2((0,T)\times\mathcal{O})} \\
&\qquad \leq c(T)\, D_\varepsilon^2\,\mathbb{E}\left\| S\left(\frac{t-s}{\varepsilon^m}\right) \mathbf{f}_{\varepsilon_n,\kappa} \right\|^2_{L^2((0,T)^2\times\mathcal{O})} \\
&\qquad \leq c\, D_\varepsilon^2\,\varepsilon^m\,\mathbb{E}\left\| S\left(\frac{-s}{\varepsilon^m}\right) \mathbf{f}_{\varepsilon_n,\kappa} \right\|^2_{L^2((0,T)\times\mathcal{O})} \\
&\qquad = c\, D_\varepsilon^2\,\varepsilon^m\,\mathbb{E}\,\|\,\mathbf{f}_{\varepsilon_n,\kappa}\,\|^2_{L^2((0,T)\times\mathcal{O})}.
\end{aligned} \tag{6.99}$$

Here $\mathbf{f}_{\varepsilon_n,\kappa}$ is comparable to the terms in $J_2, \ldots, J_4$ and we may choose

$$D_\varepsilon^2 = \max\left\{(\varepsilon^{-m}A_\varepsilon)^2, (\varepsilon^{-m}B_\varepsilon)^2, \varepsilon^{2(\alpha-m)}\right\}$$

for the entries defined in (6.87) so that given (6.63), we have that $D_\varepsilon^2 < \varepsilon$. Thus given (6.79) and (6.84), we can conclude from (6.99) and the properties of convolution that

$$\mathbb{E}\,\|J_2 + \ldots + J_4\|^2_{L^2(\mathbb{R}\times K)} \leq c(\kappa)\,\varepsilon. \tag{6.100}$$

To estimate the stochastic term, we first define the term $\hat{\Phi}_{\varepsilon_n,\kappa}(e_i) := \hat{\mathbf{g}}_i^{\varepsilon_n,\kappa} :=$

$\mathbf{g}_i\left(\cdot, \hat{\varrho}_{\varepsilon_n}(\cdot), (\hat{\mathbf{m}}_{\varepsilon_n})(\cdot)\right)_\kappa$. Then similar to the estimate for the noise term in (4.98), we can use Itô isometry, Fubini's theorem, the properties of the semigroup, Lemma 6.4.5 and the continuity of the operator $\mathcal{Q}$ to get that

$$
\begin{aligned}
\tilde{\mathbb{E}} \left\| \int_0^t S\left(\frac{t-s}{\varepsilon^m}\right) \mathcal{Q}\hat{\Phi}_{\varepsilon_n,\kappa} \, \mathrm{d}\tilde{W}_n(s) \right\|^2_{L^2((0,T)\times K)} \\
&= \tilde{\mathbb{E}} \int_0^t \sum_{i\in\mathbb{N}} \left\| S\left(\frac{t-s}{\varepsilon^m}\right) \mathcal{Q}\hat{\mathbf{g}}_i^{\varepsilon_n,\kappa} \right\|^2_{L^2((0,T)\times K)} \mathrm{d}s \\
&\leq c \int_0^T \sum_{i\in\mathbb{N}} \int_{\mathbb{R}} \tilde{\mathbb{E}} \left\| S\left(\frac{t-s}{\varepsilon^m}\right) \mathcal{Q}\hat{\mathbf{g}}_i^{\varepsilon_n,\kappa} \right\|^2_{L^2(K)} \mathrm{d}s \, \mathrm{d}t. \\
&\leq c\,\varepsilon^m \int_0^T \sum_{i\in\mathbb{N}} \tilde{\mathbb{E}} \, \|\hat{\mathbf{g}}_i^{\varepsilon_n,\kappa}\|^2_{L^2(\mathcal{O})} \, \mathrm{d}t \leq c(\kappa)\, \varepsilon^m
\end{aligned}
$$

for any $K \Subset \mathcal{O}$. We have shown the following lemma.

Lemma 6.4.6. There exists a constant c uniform in ε such that

$$
\tilde{\mathbb{E}} \, \|\hat{\varphi}_{\varepsilon_n,\kappa}(t)\|^2_{L^2((0,T)\times K)} + \tilde{\mathbb{E}} \, \|\nabla\hat{\Psi}_{\varepsilon_n,\kappa}(t)\|^2_{L^2((0,T)\times K)} \leq c(\kappa)\, \varepsilon.
$$

for any $K \Subset \mathcal{O}$.

Having shown Lemma 6.4.6, we can combine it with (4.103) for an arbitrary ball (6.44) to get (6.76), for at least a subsequence. □

To continue, we now identify the structure of the limit process for the velocity.

Lemma 6.4.7. $\tilde{\mathbb{P}}$-a.s., we have that

$$
\hat{\mathbf{u}}_{\varepsilon_n} \rightharpoonup \hat{\mathbf{U}} \quad \text{in} \quad L^2(0,T;W^{1,2}(\mathcal{O})) \tag{6.101}
$$

and with this limit being of the form

$$
\hat{\mathbf{U}} = \hat{\mathbf{U}}_h(x_h, 0) = (\hat{U}^1(x_h, 0), \hat{U}^2(x_h, 0), 0). \tag{6.102}
$$

Proof. The first part is precisely contained in Proposition 6.3.12. The important bit is showing (6.102). To see this, we observe that by the equality of laws given in Proposition 6.3.12 as well as Lemma 6.3.6, one can conclude that the sequence $\hat{\varrho}_{\varepsilon_n}$ as $n \to \infty$, converges to

$$\hat{\varrho} = 1 \tag{6.103}$$

$\tilde{\mathbb{P}}$-a.s. Because of Theorem 2.4.31, on the new probability space, one can then use this strong convergence of the density to (6.103) in order to pass to the limit in the corresponding continuity equation $(6.1)_1$ to get

$$\operatorname{div} \hat{\mathbf{U}} = 0, \quad \tilde{\mathbb{P}}\text{-a.e. in } [0,T] \times \mathcal{O} \tag{6.104}$$

which implies that

$$\partial_{x_3} \hat{U}^3 = -\partial_{x_1} \hat{U}^1 - \partial_{x_2} \hat{U}^2 \tag{6.105}$$

$\tilde{\mathbb{P}}$-a.e. in $[0,T] \times \mathcal{O}$.

Furthermore, it follows from Section 6.3.7 that $\mathcal{P}(\mathbf{e}_3 \times \hat{\mathbf{U}}) = 0$ and hence there exist a potential $\hat{\psi} = \hat{\psi}(x_h, x_3)$ such that $\mathbf{e_3} \times \hat{\mathbf{U}} = \nabla \hat{\psi}$. Or equivalently,

$$\partial_{x_1} \hat{\psi} = -\hat{U}^2, \quad \partial_{x_2} \hat{\psi} = \hat{U}^1, \quad \partial_{x_3} \hat{\psi} = 0. \tag{6.106}$$

By combining (6.105) and (6.106), we obtain

$$\partial_{x_3} \hat{U}^3 = 0$$

which implies that $\hat{U}_3 = \hat{U}_3(x_h, c)$ for a constant c independent of x_h. But since by (6.13), $\hat{U}_3(x_h, 1) = 0$, it implies that $\hat{U}_3 \equiv 0$. $\square$

6.4.8 Strong convergence for the vertical average of the solenoidal part of momentum

We now wish to control the vertical average of the solenoidal part of momentum. Since the spatial regularity of (6.53) just fails to belong to the class of Hilbert spaces, the aim is to improve this. We will require an L^2-spatial regularity in order to pass to the limit in the convective term of the momentum balance equation.
First of all, let set $\hat{\mathbf{Y}}_{\varepsilon_n,\kappa} := \mathcal{P}\left(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}\right)_\kappa$. Given (6.73), since the embedding

$$C_w\left([0,T]; L^{\frac{2\gamma}{\gamma+1}}(K)\right) \hookrightarrow L^2\left(0,T; W^{-1,2}(K)\right)$$

is continuous for $K \Subset \mathcal{O}$, we can conclude from (6.76), (6.103) and Proposition 6.3.12 that

$$\ulcorner \hat{\mathbf{Y}}_{\varepsilon_n} \urcorner \to \hat{\mathbf{U}} \quad \text{in} \quad L^2\left(0,T; W^{-1,2}(K)\right) \tag{6.107}$$

$\tilde{\mathbb{P}}$-a.s. as $n \to \infty$ (for at least a subsequence). Furthermore, for fixed $\kappa > 0$, one can find a constant $c > 0$ independent of ε_n such that

$$\left\| \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner - \hat{\mathbf{U}}_\kappa \right\|_{L^2((0,T)\times K)} \leq c(\kappa) \left\| \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n} \urcorner - \hat{\mathbf{U}} \right\|_{L^2(0,T;W^{-1,2}(K))} \tag{6.108}$$

so that we obtain

$$\ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \to \hat{\mathbf{U}}_\kappa \quad \text{in} \quad L^2\left((0,T) \times K\right) \tag{6.109}$$

$\tilde{\mathbb{P}}$-a.s. for any fixed $\kappa > 0$ as $n \to \infty$.

Remark 6.4.9. Here we remind the reader of the structure of the limit velocity (6.102).

6.4.10 Oscillatory part of momentum

As a result of compactness of the vertical average of the solenoidal part of momentum, any source of oscillation will inherently stem from the vertical coordinate dependent component of momentum . However, we can show that this oscillatory component does not interfere with the analysis of the convective term in the momentum balance equation.

Proposition 6.4.11. For all $\varepsilon_n > 0$, we let $\mathcal{P}\left(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}\right) =: \hat{\mathbf{Y}}_{\varepsilon_n}$ be the solenoidal part of momentum solving $(6.1)_2$ in the sense of distributions. Now let $\left(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\right)_{\kappa>0}$ be its regularized family obtain by convolution with the usual mollifier $\wp_\kappa$. Then for any $\boldsymbol{\phi} = [\underline{\boldsymbol{\phi}}(x_h), 0]$ with $\underline{\boldsymbol{\phi}} \in C^\infty_{c,\mathrm{div}_h}(\mathbb{R}^2)$ and $t \in [0,T]$, we have that

$$-\lim_{\varepsilon_n \to 0} \int_0^t \left\langle \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\,,\, \nabla \boldsymbol{\phi} \right\rangle \mathrm{d}\tau = \int_0^t \int_{\mathbb{R}^2} \mathrm{div}\left(\hat{\mathbf{U}}_\kappa \otimes \hat{\mathbf{U}}_\kappa\right) \cdot \boldsymbol{\phi}\, \mathrm{d}x_h\, \mathrm{d}\tau$$

holds $\tilde{\mathbb{P}}$-a.s.

Proof. For the following decomposition

$$\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}(x) = \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner (x_h) + \llcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \lrcorner (x), \tag{6.110}$$

we observe that

$$\ulcorner \llcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \lrcorner \urcorner = 0 \tag{6.111}$$

a.s. and as such, we can find a $I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)$ such that

$$\llcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \lrcorner = \partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big), \quad \ulcorner I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \urcorner = 0 \tag{6.112}$$

c.f. [36, Section 3.3.2] and [51, Section 3.2].

Let now rewrite (6.82) and mollify the resultant system to obtain the following

$$
\begin{aligned}
\varepsilon\, \mathrm{d}\big[\eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa + \mathbf{e}_3 \times \big[\eta_\varepsilon\big(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\big)\big]_\kappa \,\mathrm{d}t &= \varepsilon^{m-1}\big[\eta_\varepsilon(\hat{r}_{\varepsilon_n}\nabla G)\big]_\kappa \,\mathrm{d}t \\
&+ \varepsilon \operatorname{div}\big[\eta_\varepsilon \mathbb{S}(\nabla \hat{\mathbf{u}}_{\varepsilon_n}) - \eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa \mathrm{d}t \\
&- \varepsilon \left[\nabla \eta_\varepsilon \cdot \big(\mathbb{S}(\nabla \hat{\mathbf{u}}_{\varepsilon_n}) - (\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n})\big)\right]_\kappa \mathrm{d}t \\
&+ \varepsilon^{1-2m}\Big(\big[\nabla \eta_\varepsilon\,(\hat{\varrho}^\gamma_{\varepsilon_n} - \overline{\varrho}^\gamma_\varepsilon)\big]_\kappa - \nabla\big[\eta_\varepsilon\big(\hat{\varrho}^\gamma_{\varepsilon_n} - \overline{\varrho}^\gamma_\varepsilon\big)\big]_\kappa\Big)\mathrm{d}t \\
&+ \varepsilon \left[\eta_\varepsilon\, \Phi(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\right]_\kappa \mathrm{d}\tilde{W}_{\varepsilon_n} \\
&=: \varepsilon\, \mathbf{Q}_{\varepsilon_n,\kappa}\,\mathrm{d}t + \varepsilon^{1-2m}\mathbf{P}_{\varepsilon_n,\kappa}\,\mathrm{d}t + \varepsilon\, \Phi_{\varepsilon_n,\kappa}\mathrm{d}\tilde{W}_{\varepsilon_n}
\end{aligned}
\tag{6.113}
$$

where

$$
\begin{aligned}
\mathbf{Q}_{\varepsilon_n,\kappa} &:= \varepsilon^{m-2}\big[\eta_\varepsilon(\hat{r}_{\varepsilon_n}\nabla G)\big]_\kappa \\
&\quad + \operatorname{div}\big[\eta_\varepsilon \mathbb{S}(\nabla \hat{\mathbf{u}}_{\varepsilon_n}) - \eta_\varepsilon(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n})\big]_\kappa \\
&\quad - \left[\nabla \eta_\varepsilon \cdot \big(\mathbb{S}(\nabla \hat{\mathbf{u}}_{\varepsilon_n}) - (\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n})\big)\right]_\kappa, \\
\mathbf{P}_{\varepsilon_n,\kappa} &:= \Big(\big[\nabla \eta_\varepsilon\,(\hat{\varrho}^\gamma_{\varepsilon_n} - \overline{\varrho}^\gamma_\varepsilon)\big]_\kappa - \nabla\big[\eta_\varepsilon\big(\hat{\varrho}^\gamma_{\varepsilon_n} - \overline{\varrho}^\gamma_\varepsilon\big)\big]_\kappa\Big), \\
\Phi_{\varepsilon_n,\kappa} &:= \varepsilon\left[\eta_\varepsilon\, \Phi(\hat{\varrho}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\right]_\kappa.
\end{aligned}
\tag{6.114}
$$

Furthermore, rewriting (6.113) in component form and differentiating results in the following

$$
\begin{aligned}
\varepsilon\, \mathrm{d}\partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n}\hat{u}_{\varepsilon_n}\big)^1_\kappa - \partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n}\hat{u}_{\varepsilon_n}\big)^2_\kappa \,\mathrm{d}t &= \varepsilon\, \partial_{x_i} Q^1_{\varepsilon_n,\kappa}\,\mathrm{d}t \\
&+ \varepsilon^{1-2m}\, \partial_{x_i} P^1_{\varepsilon_n,\kappa}\,\mathrm{d}t + \varepsilon\, \partial_{x_i}\Phi^1_{\varepsilon_n,\kappa}\mathrm{d}\tilde{W}_{\varepsilon_n}
\end{aligned}
\tag{6.115}
$$

$$
\begin{aligned}
\varepsilon\, \mathrm{d}\partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n}\hat{u}_{\varepsilon_n}\big)^2_\kappa + \partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n}\hat{u}_{\varepsilon_n}\big)^1_\kappa \,\mathrm{d}t &= \varepsilon\, \partial_{x_i} Q^2_{\varepsilon_n,\kappa}\,\mathrm{d}t \\
&+ \varepsilon^{1-2m}\, \partial_{x_i} P^2_{\varepsilon_n,\kappa}\,\mathrm{d}t + \varepsilon\, \partial_{x_i}\Phi^2_{\varepsilon_n,\kappa}\mathrm{d}\tilde{W}_{\varepsilon_n}
\end{aligned}
\tag{6.116}
$$

$$
\begin{aligned}
\varepsilon\, \mathrm{d}\partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n}\hat{u}_{\varepsilon_n}\big)^3_\kappa &= \varepsilon\, \partial_{x_i} Q^3_{\varepsilon_n,\kappa}\,\mathrm{d}t \\
&+ \varepsilon^{1-2m}\, \partial_{x_i} P^3_{\varepsilon_n,\kappa}\,\mathrm{d}t + \varepsilon\, \partial_{x_i}\Phi^3_{\varepsilon_n,\kappa}\mathrm{d}\tilde{W}_{\varepsilon_n}
\end{aligned}
\tag{6.117}
$$

for $i = 1, 2, 3$.

We now recall that in terms of coordinates, we can write the decomposition of momentum into its solenoidal and gradient parts as

$$\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{u}_{\varepsilon_n}\big)^i_\kappa = \hat{Y}^i_{\varepsilon_n,\kappa} + \partial_{x_i} \hat{\Psi}_{\varepsilon_n,\kappa}, \quad i = 1,2,3. \tag{6.118}$$

As such, the symmetry of the Hessian means that we can define the traceless skew-symmetric operator

$$\hat{\Upsilon}^{ij}_{\varepsilon_n,\kappa} := \partial_{x_i}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{u}_{\varepsilon_n}\big)^j_\kappa - \partial_{x_j}\big(\eta_\varepsilon \hat{\varrho}_{\varepsilon_n} \hat{u}_{\varepsilon_n}\big)^i_\kappa = \partial_{x_i} \hat{Y}^j_{\varepsilon_n,\kappa} - \partial_{x_j} \hat{Y}^i_{\varepsilon_n,\kappa} \tag{6.119}$$

for $i,j = 1,2,3$.

By using the relation (6.119), we get that (6.117) with $i = 2$ minus (6.116) with $i = 3$ yields

$$\varepsilon \mathrm{d}\, \hat{\Upsilon}^{23}_{\varepsilon_n,\kappa} - \partial_{x_3} \hat{Y}^1_{\varepsilon_n,\kappa} \mathrm{d}t = \partial_{x_3 x_1} \hat{\Psi}_{\varepsilon_n,\kappa} \mathrm{d}t + \varepsilon Q^{23}_{\varepsilon_n,\kappa} \mathrm{d}t + \varepsilon \Phi^{23}_{\varepsilon_n,\kappa}\, \mathrm{d}\tilde{W}_{\varepsilon_n} \tag{6.120}$$

having used the additional decomposition (6.118). In analogy with (6.119),

$$Q^{ij}_{\varepsilon_n,\kappa} := \partial_{x_i} Q^j_{\varepsilon_n,\kappa} - \partial_{x_j} Q^i_{\varepsilon_n,\kappa}$$

with an identical notation for the noise term.

Similar to (6.120), (6.117) with $i = 1$ minus (6.115) with $i = 3$ yields

$$\varepsilon \mathrm{d}\, \hat{\Upsilon}^{13}_{\varepsilon_n,\kappa} + \partial_{x_3} \hat{Y}^2_{\varepsilon_n,\kappa} \mathrm{d}t = -\partial_{x_3 x_2} \hat{\Psi}_{\varepsilon_n,\kappa} \mathrm{d}t + \varepsilon Q^{13}_{\varepsilon_n,\kappa} \mathrm{d}t + \varepsilon \Phi^{13}_{\varepsilon_n,\kappa}\, \mathrm{d}\tilde{W}_{\varepsilon_n}. \tag{6.121}$$

Lastly, (6.116) with $i = 1$ minus (6.115)with $i = 2$ gives

$$\varepsilon \mathrm{d}\, \hat{\Upsilon}^{12}_{\varepsilon_n,\kappa} + \mathrm{div}_h \big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h \mathrm{d}t = -\Delta_h \hat{\Psi}_{\varepsilon_n,\kappa}\, \mathrm{d}t + \varepsilon Q^{12}_{\varepsilon_n,\kappa} \mathrm{d}t + \varepsilon \Phi^{12}_{\varepsilon_n,\kappa}\, \mathrm{d}\tilde{W}_{\varepsilon_n} \tag{6.122}$$

by the use of (6.118) above.

Now using the fact that $\operatorname{div} \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} = 0$, we have that

$$\begin{aligned}
\operatorname{div}\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) &= \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \cdot \nabla \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \\
&= \frac{1}{2}\nabla \left|\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\right|^2 - \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \times \operatorname{curl}\left(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\right).
\end{aligned} \tag{6.123}$$

Furthermore, by linearity and commutativity of the curl and derivative operators, we can use (6.110)–(6.112) to get

$$\begin{aligned}
\operatorname{div}\left(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\right) &= \frac{1}{2}\nabla \left|\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\right|^2 - \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \times \operatorname{curl} \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \\
&- \partial_{x_3}\left[I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \operatorname{curl} \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner + \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \times \operatorname{curl}\left(I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\right)\right] \\
&- \partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \operatorname{curl}\left(\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\right)
\end{aligned} \tag{6.124}$$

where

$$\ulcorner \partial_{x_3}\left[I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \operatorname{curl} \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner + \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \times \operatorname{curl}\left(I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\right)\right] \urcorner = 0. \tag{6.125}$$

With (6.125) in hand, we wish to show that the last term in (6.124) above converges to zero in a suitable sense. To see this, we perform a direct computation using (6.119) to gain

$$\begin{aligned}
&\left[\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \operatorname{curl}\left(\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\right)\right]^1 \\
&\quad = \big(\partial_{x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big)\partial_{x_3}\left[\partial_{x_1} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big) - \partial_{x_2} I\big(\hat{Y}^1_{\varepsilon_n,\kappa}\big)\right] \\
&\quad - \big(\partial_{x_3} I\big(\hat{Y}^3_{\varepsilon_n,\kappa}\big)\big)\partial_{x_3}\left[\partial_{x_3} I\big(\hat{Y}^1_{\varepsilon_n,\kappa}\big) - \partial_{x_1} I\big(\hat{Y}^3_{\varepsilon_n,\kappa}\big)\right] \\
&\quad = \big(\partial_{x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big)\partial_{x_3} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big) - \big(\partial_{x_3} I\big(\hat{Y}^3_{\varepsilon_n,\kappa}\big)\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big) \\
&\quad = \partial_{x_3}\left[\big(\partial_{x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\right] - \big(\partial_{x_3x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big) \\
&\quad + I\big(\operatorname{div}_h \big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)
\end{aligned} \tag{6.126}$$

where we have used $\operatorname{div}\big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big] = 0$ and hence $\operatorname{div}_h \big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h = -\partial_{x_3} \hat{Y}^3_{\varepsilon_n,\kappa}$ in the last step above.

Since the vertical average of the first term on the right-hand side of (6.126) vanishes, we concentrate on the last two terms. For this, we first use (6.121) to formally obtain

$$
\begin{aligned}
-\varepsilon\Big[\mathrm{d}\,\partial_{x_3} I\big(\hat{\Upsilon}^{13}_{\varepsilon_n,\kappa}\big)\Big] I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big) &= \big(\partial_{x_3x_3} I\big(\hat{Y}^{2}_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t \\
&+ \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t - \varepsilon\big(Q^{13}_{\varepsilon_n,\kappa} - \ulcorner \partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&- \varepsilon\big(\Phi^{13}_{\varepsilon_n,\kappa} - \ulcorner \partial_{x_1}\Phi^{3}_{\varepsilon_n,\kappa}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{W}_{\varepsilon_n}
\end{aligned}
\tag{6.127}
$$

since

$$
\begin{aligned}
\partial_{x_3} I\Big[\partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa}\Big] &= \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} - \partial_{x_3}\ulcorner\partial_{x_2}\hat{\Psi}_{\varepsilon_n,\kappa}\urcorner = \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} \\
\partial_{x_3} I\big[Q^{13}_{\varepsilon_n,\kappa}\big] &= Q^{13}_{\varepsilon_n,\kappa} - \ulcorner\partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner + \partial_{x_3}\ulcorner Q^{1}_{\varepsilon_n,\kappa}\urcorner \\
&= Q^{13}_{\varepsilon_n,\kappa} - \ulcorner\partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner
\end{aligned}
$$

and similarly for the noise term.

By using (6.122), we also formally obtain

$$
\begin{aligned}
-\varepsilon\big[\mathrm{d}\, I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\big]\partial_{x_3} I\big(\hat{\Upsilon}^{13}_{\varepsilon_n,\kappa}\big) &= \varepsilon\big[\mathrm{d}\, I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\big]\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big) \\
= -I\big(\mathrm{div}_h\big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t &- I\big(\Delta_h\hat{\Psi}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t \\
+ \varepsilon I\big(Q^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t &+ \varepsilon I\big(\Phi^{12}_{\varepsilon_n,\kappa}\big)\,\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{W}_{\varepsilon_n}.
\end{aligned}
\tag{6.128}
$$

Now by Iˆto's formula, Theorem 2.4.37, it follows from (6.127) and (6.128) that

$$
\begin{aligned}
-\varepsilon\mathrm{d}\,\big[I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{13}_{\varepsilon_n,\kappa}\big)\big] &= -I\big(\mathrm{div}_h\big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t \\
&- I\big(\Delta_h\hat{\Psi}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t + \varepsilon I\big(Q^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&+ \big(\partial_{x_3x_3} I\big(\hat{Y}^{2}_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t + \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&- \varepsilon\big(Q^{13}_{\varepsilon_n,\kappa} - \ulcorner\partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t + \varepsilon\sum_{k\in\mathbb{N}} I\big(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k}\big)\,\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&- \varepsilon\sum_{k\in\mathbb{N}}\big(\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner\partial_{x_1}\mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&- \varepsilon\sum_{k\in\mathbb{N}} I(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k})\big[\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner\partial_{x_1}\mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big]\,\mathrm{d}t
\end{aligned}
\tag{6.129}
$$

where

$$
\mathbf{g}^{i}_{\varepsilon_n,\kappa,k} = [\Phi_{\varepsilon_n,\kappa}(e_k)]^i, \quad \mathbf{g}^{ij}_{\varepsilon_n,\kappa,k} = \partial_{x_i}[\Phi_{\varepsilon_n,\kappa}(e_k)]^j - \partial_{x_j}[\Phi_{\varepsilon_n,\kappa}(e_k)]^i.
$$

We can now rearrange (6.129) to get

$$\begin{aligned}
&-\big(\partial_{x_3x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t + I\big(\mathrm{div}_h\big[\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big]_h\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t \\
&\quad = \varepsilon \mathrm{d}\left[I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{13}_{\varepsilon_n,\kappa}\big)\right] + \varepsilon I\big(Q^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&\quad - \varepsilon\big(Q^{13}_{\varepsilon_n,\kappa} - \ulcorner \partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t + \varepsilon \sum_{k\in\mathbb{N}} I\big(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k}\big)\,\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} \big(\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_1} \mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} I(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k})\big[\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_1} \mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big]\,\mathrm{d}t \\
&\quad - I\big(\Delta_h \hat{\Psi}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t + \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t.
\end{aligned} \tag{6.130}$$

Substituting (6.127)–(6.130) into (6.126), we have shown that

$$\begin{aligned}
&\left[\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \mathrm{curl}\left(\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\right)\right]^1 \mathrm{d}t = \partial_{x_3}\left[\big(\partial_{x_3} I\big(\hat{Y}^2_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\right]\mathrm{d}t \\
&\quad + \varepsilon \mathrm{d}\left[I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{13}_{\varepsilon_n,\kappa}\big)\right] + \varepsilon I\big(Q^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&\quad - \varepsilon\big(Q^{13}_{\varepsilon_n,\kappa} - \ulcorner \partial_{x_1} Q^{3}_{\varepsilon_n,\kappa}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t + \varepsilon \sum_{k\in\mathbb{N}} I\big(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k}\big)\,\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} \big(\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_1} \mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} I(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k})\big[\mathbf{g}^{13}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_1} \mathbf{g}^{3}_{\varepsilon_n,\kappa,k}\urcorner\big]\,\mathrm{d}t \\
&\quad - I\big(\Delta_h \hat{\Psi}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{31}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t + \partial_{x_3x_2}\hat{\Psi}_{\varepsilon_n,\kappa} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&\quad =: J_1 + \ldots + J_9.
\end{aligned} \tag{6.131}$$

Similarly, we gain

$$
\begin{aligned}
&\Big[\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big) \times \operatorname{curl}\Big(\partial_{x_3} I\big(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\big)\Big)\Big]^2 \mathrm{d}t = \partial_{x_3}\Big[\big(\partial_{x_3} I\big(\hat{Y}^1_{\varepsilon_n,\kappa}\big)\big) I\big(\hat{\Upsilon}^{21}_{\varepsilon_n,\kappa}\big)\Big]\,\mathrm{d}t \\
&\quad + \varepsilon \mathrm{d}\left[I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{23}_{\varepsilon_n,\kappa}\big)\right] + \varepsilon I\big(Q^{12}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{23}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&\quad - \varepsilon\big(Q^{23}_{\varepsilon_n,\kappa} - \ulcorner \partial_{x_2} Q^3_{\varepsilon_n,\kappa} \urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t + \varepsilon \sum_{k\in\mathbb{N}} I\big(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k}\big)\,\partial_{x_3} I\big(\hat{\Upsilon}^{23}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} \big(\mathbf{g}^{23}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_2} \mathbf{g}^3_{\varepsilon_n,\kappa,k} \urcorner\big) I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}\tilde{\beta}^{\varepsilon_n}_k \qquad (6.132) \\
&\quad - \varepsilon \sum_{k\in\mathbb{N}} I(\mathbf{g}^{12}_{\varepsilon_n,\kappa,k})\big[\mathbf{g}^{23}_{\varepsilon_n,\kappa,k} - \ulcorner \partial_{x_2} \mathbf{g}^3_{\varepsilon_n,\kappa,k} \urcorner\big]\,\mathrm{d}t \\
&\quad + I\big(\Delta_h \hat{\Psi}_{\varepsilon_n,\kappa}\big)\partial_{x_3} I\big(\hat{\Upsilon}^{23}_{\varepsilon_n,\kappa}\big)\,\mathrm{d}t - \partial_{x_3 x_1}\hat{\Psi}_{\varepsilon_n,\kappa} I\big(\hat{\Upsilon}^{12}_{\varepsilon_n,\kappa}\big)\mathrm{d}t \\
&\quad =: K_1 + \ldots + K_9
\end{aligned}
$$

where both J_1 and K_1 vanishes after the taking of vertical averages. Furthermore, given that by Proposition 6.3.12, the smoothness of $\mathbf{g}_{\varepsilon_n,\kappa,k}$ is preserved and that the terms $J_2, \ldots, J_7$ and $K_2, \ldots, K_7$ are smooth and hence uniformly bounded in ε_n in suitable Bochner spaces, recall Section 6.3, we are left to worry about the terms J_8, J_9, K_8 and K_9. This is because, per the explanation for $J_2, \ldots, J_7$ and $K_2, \ldots, K_7$ above, we obtain that for any $\boldsymbol{\phi} = [\underline{\boldsymbol{\phi}}(x_h), 0]$ with $\underline{\boldsymbol{\phi}} \in C^\infty_{c,\mathrm{div}_h}(\mathbb{R}^2)$, any $\psi \in L^2(\Omega \times (0,t))$ and any $t \in [0,T]$,

$$
\begin{aligned}
\tilde{\mathbb{E}} \int_0^t \big\langle J_i\,,\, \boldsymbol{\phi}\,\psi \big\rangle\,\mathrm{d}\tau &\lesssim \varepsilon \big\| J_i \big\|_{L^{p_1}_\omega L^{p_2}_t L^{p_3}_x} \big\|\boldsymbol{\phi}\big\|_{L^q_x} \big\|\psi\big\|_{L^2_{\omega,t}} \\
&\lesssim \varepsilon \to 0
\end{aligned} \qquad (6.133)
$$

for suitable $p_i \geq 1$ and $q \geq 1$ as $\varepsilon \to 0$. The same applies for the K_i's with $i = 2, \ldots, 7$. The noise terms follow in a similar manner by the use of Itô isometry and the fact that squared-integrable functions (in probability) are integrable.

For these extra terms J_8, J_9, K_8 and K_9, we observe that for any $t \in (0,T)$ and any $i,j = 1,2,3$,

$$
\begin{aligned}
\int_0^t \|\partial_{x_j x_i} \hat{\Psi}_{\varepsilon_n,\kappa}\|^2_{L^2(K)}\,\mathrm{d}t &\lesssim \int_0^T \|\partial_{x_i} \wp_\kappa\|^2_{L^{\frac{2\gamma}{2\gamma-1}}(K)} \|\nabla \hat{\Psi}_{\varepsilon_n}\|^2_{L^{\frac{2\gamma}{\gamma+1}}(K)}\,\mathrm{d}t \\
&\lesssim_\kappa \int_0^T \|\mathcal{Q}(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})\|^2_{L^{\frac{2\gamma}{\gamma+1}}(K)}\,\mathrm{d}t \to 0
\end{aligned} \qquad (6.134)
$$

$\tilde{\mathbb{P}}$-a.s. for $K \Subset \mathcal{O}$ as $n \to \infty$. This follows by (6.76) hence terms J_8, J_9, K_8 and K_9 also vanishes in the limit.

We can now take the vertical average in (6.123) and combine it with (6.124), (6.131), (6.132) (keeping in mind, the argument after (6.132)) and (6.134) to get that for any $\boldsymbol{\phi} = [\underline{\boldsymbol{\phi}}(x_h), 0]$ with $\underline{\boldsymbol{\phi}} \in C^\infty_{c,\mathrm{div}_h}(\mathbb{R}^2)$, any $\psi \in L^2(\Omega \times (0,t))$ and $t \in [0,T]$ that

$$\begin{aligned}
&-\lim_{n\to\infty} \tilde{\mathbb{E}} \int_0^t \left\langle \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \,,\, \nabla \boldsymbol{\phi}\, \psi \right\rangle \mathrm{d}\tau \\
&= \lim_{n\to\infty} \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^2} \ulcorner \mathrm{div} \left(\hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \right) \urcorner \cdot \boldsymbol{\phi}\, \psi \, \mathrm{d}x_h \, \mathrm{d}\tau \\
&= \lim_{n\to\infty} \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^2} \ulcorner \left[\frac{1}{2} \nabla \left| \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \right|^2 - \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \times \mathrm{curl} \ulcorner \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \urcorner \right] \urcorner \cdot \boldsymbol{\phi}\, \psi \, \mathrm{d}x_h \mathrm{d}\tau \\
&= \tilde{\mathbb{E}} \int_0^t \int_{\mathbb{R}^2} \mathrm{div} \left(\hat{\mathbf{U}}_\kappa \otimes \hat{\mathbf{U}}_\kappa \right) \cdot \boldsymbol{\phi}\, \psi \, \mathrm{d}x_h \, \mathrm{d}\tau.
\end{aligned} \tag{6.135}$$

This is because the gradient term vanishes after integration by part whereas convergence to velocity holds for the vertical average of the solenoidal part of momentum. cf. (6.109). Subsequently, since ψ is arbitrary, we gain

$$\begin{aligned}
-\lim_{n\to\infty} \int_0^t \left\langle \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \,,\, \nabla \boldsymbol{\phi} \right\rangle \mathrm{d}\tau \\
= \int_0^t \int_{\mathbb{R}^2} \mathrm{div} \left(\hat{\mathbf{U}}_\kappa \otimes \hat{\mathbf{U}}_\kappa \right) \cdot \boldsymbol{\phi} \, \mathrm{d}x_h \, \mathrm{d}\tau
\end{aligned} \tag{6.136}$$

$\tilde{\mathbb{P}}$-a.s. possibly for a further subsequence. $\square$

6.5 Conclusion

We now wish to apply the 2-D uniqueness Theorem 6.2.12 to obtain convergence of our original sequence to a limit random variable that will solve (6.4) in the pathwise sense given by Definition 6.2.11. To do this however, we first identify the limit of the convective term before finally, completing the proof of our main theorem.

6.5.1 Identifying the convective term

We show in this section that in identifying the limit of the convective term, we may essentially replace the sequence $(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n})$ by the mollified term $(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})_\kappa\otimes(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n})_\kappa$ since their limits coincide. This is given in the statement of the lemma below.

Lemma 6.5.2. For any $\boldsymbol{\phi}=[\underline{\boldsymbol{\phi}}(x_h),0]$ with $\underline{\boldsymbol{\phi}}\in C^\infty_{c,\mathrm{div}_h}(\mathbb{R}^2)$ and for all $t\in[0,T]$, the convergence

$$\lim_{\varepsilon_n\to 0}\int_0^t\langle\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n}\,,\,\nabla\boldsymbol{\phi}\rangle\,\mathrm{d}\tau=\int_0^t\left\langle\hat{\mathbf{U}}\otimes\hat{\mathbf{U}}\,,\,\nabla\boldsymbol{\phi}\right\rangle\,\mathrm{d}\tau \tag{6.137}$$

holds $\tilde{\mathbb{P}}$-a.s.

Proof. To show (6.137), we first make the following decomposition[3]

$$\begin{aligned}
\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n}&=[(1-\hat{\varrho}_{\varepsilon_n})\hat{\mathbf{u}}_{\varepsilon_n}]_\kappa\otimes[(\hat{\varrho}_{\varepsilon_n}-1)\hat{\mathbf{u}}_{\varepsilon_n}]_\kappa\\
&+(\hat{\varrho}_{\varepsilon_n}-1)\hat{\mathbf{u}}_{\varepsilon_n}\otimes\hat{\mathbf{u}}_{\varepsilon_n}+[(1-\hat{\varrho}_{\varepsilon_n})\hat{\mathbf{u}}_{\varepsilon_n}]_\kappa\otimes\hat{\mathbf{u}}_{\varepsilon_n,\kappa}\\
&+\hat{\mathbf{u}}_{\varepsilon_n,\kappa}\otimes[(1-\hat{\varrho}_{\varepsilon_n})\hat{\mathbf{u}}_{\varepsilon_n}]_\kappa+\hat{\mathbf{u}}_{\varepsilon_n,\kappa}\otimes(\hat{\mathbf{u}}_{\varepsilon_n}-\hat{\mathbf{u}}_{\varepsilon_n,\kappa})\\
&+\left(\hat{\mathbf{u}}_{\varepsilon_n}-\hat{\mathbf{u}}_{\varepsilon_n,\kappa}\right)\otimes\hat{\mathbf{u}}_{\varepsilon_n}+\left(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\right)_\kappa\otimes\left(\hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n}\right)_\kappa\\
&=:\sum_{i=1}^{7}J_i.
\end{aligned} \tag{6.138}$$

Then we observe that for any ball B_k of radius $k>0$ and fixed regularizing kernel $\kappa>0$,

$$J_i\to 0\quad\text{in}\quad L^1((0,T)\times B_k)$$

$\tilde{\mathbb{P}}$-a.s. for each $i\in\{1,2,3,4\}$ as $n\to\infty$. This follows from Lemma 6.3.6 and the equality of laws given by Proposition 6.3.12.

Now we notice that for the random variable $\hat{\mathbf{u}}_{\varepsilon_n}\in L^2(0,T;W^{1,2}(B_k))$, one can find

[3] This decomposition is not unique and in fact, a much simpler variant suffices.

a constant $c > 0$ independent of both κ and ε_n such that

$$\|\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\|_{L^2(\Omega;L^2(0,T;L^p(B_k)))} \leq c\,\kappa^{3/p-1/2} \tag{6.139}$$

holds uniformly in $\varepsilon_n > 0$ for all $p \in [2,6)$, cf. [51, Section 3.2]. It follows from a density argument and the Riesz representation theorem for integrable functions that for all $\boldsymbol{\phi}(x) \in C_c^\infty(\mathcal{O})$ and all $\boldsymbol{\phi}_1(\omega) \in L^\infty(\Omega)$, $\boldsymbol{\phi}_2(t) \in L^\infty(0,T)$, one can find a generic constant $c > 0$ that is uniform in both ε_n and κ such that

$$\begin{aligned}
&\left\|\left\langle \hat{\mathbf{u}}_{\varepsilon_n,\kappa} \otimes \left(\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\right), \nabla\boldsymbol{\phi}\right\rangle\right\|_{L^1_{\omega,t}} \\
&= \sup_{\|\boldsymbol{\phi}_1\boldsymbol{\phi}_2\|_\infty \leq 1} \tilde{\mathbb{E}}\int_0^T \left\langle \hat{\mathbf{u}}_{\varepsilon_n,\kappa} \otimes \left(\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\right), \nabla\boldsymbol{\phi}\right\rangle \boldsymbol{\phi}_1(\omega)\boldsymbol{\phi}_2(t)\,\mathrm{d}t \\
&\leq c\,\|\nabla\boldsymbol{\phi}\|_{L^2_x}\|\hat{\mathbf{u}}_{\varepsilon_n,\kappa}\|_{L^2_{\omega,t}L^6_x}\,\|\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\|_{L^2_{\omega,t}L^3_x} \\
&\leq c\,\|\hat{\mathbf{u}}_{\varepsilon_n}\|_{L^2_{\omega,t}L^6_x}\,\|\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\|^{1/2}_{L^2_{\omega,t}W^{1,2}_x}\,\|\hat{\mathbf{u}}_{\varepsilon_n} - \hat{\mathbf{u}}_{\varepsilon_n,\kappa}\|^{1/2}_{L^2_{\omega,t}L^2_x} \\
&\leq c\,\sqrt{\kappa}\,\|\hat{\mathbf{u}}_{\varepsilon_n}\|_{L^2_{\omega,t}W^{1,2}_x} \leq c\,\sqrt{\kappa}.
\end{aligned} \tag{6.140}$$

It follows that as $n \to \infty$ and $\kappa \to 0$, $\int_0^T \langle J_5\,,\,\nabla\boldsymbol{\phi}\rangle\,\mathrm{d}t$ vanishes in mean and hence in probability (after taking a subsequence). The same argument holds for J_6.

As a consequence of (6.76), (6.136), (6.138) and the above convergence to zero results, one has that

$$\begin{aligned}
\lim_{\varepsilon_n \to 0}\int_0^t &\left\langle \hat{\varrho}_{\varepsilon_n}\hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n}\,,\,\nabla\boldsymbol{\phi}\right\rangle \mathrm{d}\tau \\
&= \lim_{\kappa\to 0}\lim_{\varepsilon_n\to 0}\int_0^t \left\langle \hat{\mathbf{Y}}_{\varepsilon_n,\kappa} \otimes \hat{\mathbf{Y}}_{\varepsilon_n,\kappa}\,,\,\nabla\boldsymbol{\phi}\right\rangle \mathrm{d}\tau \\
&= \int_0^t \left\langle \hat{\mathbf{U}} \otimes \hat{\mathbf{U}}\,,\,\nabla\boldsymbol{\phi}\right\rangle \mathrm{d}\tau
\end{aligned} \tag{6.141}$$

for any $t \in (0,T]$. □

Proof of Theorem 6.3.14. From Lemma 6.3.15, $\tilde{W}_n$ are cylindrical Wiener processes and as such, we can find a collection of mutually independent 1-D $(\tilde{\mathscr{F}}_t)$-Brownian motions $(\tilde{\beta}_k)_{k\in\mathbb{N}}$ and orthonormal basis $(e_k)_{k\in\mathbb{N}}$ such that $\tilde{W} = \sum_{k\in\mathbb{N}}\tilde{\beta}_k e_k$. We refer the reader to Lemma 4.7.3 above for further details.

To show that $[(\tilde{\Omega}, \tilde{\mathscr{F}}_t, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \hat{\mathbf{U}}, \tilde{W}]$ satisfies (6.4) in the distributional sense, we consider the functionals

$$\begin{aligned} M(\rho, \mathbf{u}, \mathbf{m})_t &= \langle \mathbf{m}(t), \boldsymbol{\phi} \rangle - \langle \mathbf{m}(0), \boldsymbol{\phi} \rangle - \int_0^t \langle \mathbf{m} \otimes \mathbf{u} - \nu \nabla \mathbf{u}, \nabla \boldsymbol{\phi} \rangle \mathrm{d}r \\ &\quad + (\lambda + \nu) \int_0^t \langle \operatorname{div} \mathbf{u}, \operatorname{div} \boldsymbol{\phi} \rangle \mathrm{d}r - \frac{1}{\varepsilon^{2m}} \int_0^t \langle \rho^\gamma, \operatorname{div} \boldsymbol{\phi} \rangle \mathrm{d}r \\ &\quad + \frac{1}{\varepsilon} \int_0^t \langle \mathbf{e}_3 \times \mathbf{m},\, \boldsymbol{\phi} \rangle \mathrm{d}r - \frac{1}{\varepsilon^2} \int_0^t \langle \rho \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r, \\ N(\rho, \mathbf{m})_t &= \sum_{k \in \mathbb{N}} \int_0^t \langle \mathbf{g}_k(\rho, \mathbf{m})\,,\, \boldsymbol{\phi} \rangle^2 \, \mathrm{d}r, \\ N_k(\rho, \mathbf{m})_t &= \int_0^t \langle \mathbf{g}_k(\rho, \mathbf{m})\,,\, \boldsymbol{\phi} \rangle \, \mathrm{d}r, \end{aligned}$$

for all $\boldsymbol{\phi} = (\underline{\boldsymbol{\phi}}, 0) \in C^\infty_{c,\mathrm{div}}(\mathcal{O})$ where $\underline{\boldsymbol{\phi}} \in C^\infty_{c,\mathrm{div}_h}(\mathbb{R}^2)$. Then by Section 6.3.7 and integration by parts, we have that

$$\frac{1}{\varepsilon} \int_0^t \langle \mathbf{e}_3 \times \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n},\, \boldsymbol{\phi} \rangle \mathrm{d}r = \frac{1}{\varepsilon} \int_0^t \big\langle \ulcorner \mathbf{e}_3 \times \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n} \urcorner,\, \underline{\boldsymbol{\phi}} \big\rangle_h \mathrm{d}r = 0. \tag{6.142}$$

Also since $\operatorname{div} \boldsymbol{\phi} = 0$, we can use (6.16) and $\hat{r}_{\varepsilon_n} = \frac{\hat{\varrho}_{\varepsilon_n} - \overline{\varrho}_\varepsilon}{\varepsilon^m}$ to show that

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_0^t \langle \hat{\varrho}_{\varepsilon_n} \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r &= \frac{1}{\varepsilon^2} \int_0^t \langle \overline{\varrho}_\varepsilon \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r \\ &\quad + \frac{1}{\varepsilon^2} \int_0^t \langle (\hat{\varrho}_{\varepsilon_n} - \overline{\varrho}_\varepsilon) \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r \\ &= \frac{1}{\varepsilon^{2m}} \int_0^t \langle \nabla \overline{\varrho}^\gamma_\varepsilon\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r + \varepsilon^{m-2} \int_0^t \langle \hat{r}_{\varepsilon_n} \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r \\ &= \varepsilon^{m-2} \int_0^t \langle \hat{r}_{\varepsilon_n} \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r \end{aligned} \tag{6.143}$$

by integration by parts. Finally since $\langle \operatorname{div} \hat{\mathbf{u}}_{\varepsilon_n}\,, \operatorname{div} \boldsymbol{\phi} \rangle = \langle \hat{\varrho}^\gamma_{\varepsilon_n}\,, \operatorname{div} \boldsymbol{\phi} \rangle = 0$, we can therefore conclude that for $m > 10$,

$$\begin{aligned} M\big(\hat{\varrho}_{\varepsilon_n}, \hat{\mathbf{u}}_{\varepsilon_n}, \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}\big)_t &= \big\langle \big(\hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}\big)(t), \boldsymbol{\phi} \big\rangle - \big\langle \big(\hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n}\big)(0), \boldsymbol{\phi} \big\rangle \\ - \int_0^t \langle \hat{\varrho}_{\varepsilon_n} \hat{\mathbf{u}}_{\varepsilon_n} \otimes \hat{\mathbf{u}}_{\varepsilon_n} &- \nu \nabla \hat{\mathbf{u}}_{\varepsilon_n}, \nabla \boldsymbol{\phi} \rangle \mathrm{d}r + \varepsilon^{m-2} \int_0^t \langle \hat{r}_{\varepsilon_n} \nabla G\,,\, \boldsymbol{\phi} \rangle \mathrm{d}r. \end{aligned}$$

As a result of (6.58) and Proposition 6.3.12, we can conclude that the last term above vanishes $\tilde{\mathbb{P}}$-a.s. in the limit as $\varepsilon \to 0$.

Now passing to the limit $n \to \infty$ in the function $M(\cdot)_t$ and keeping in mind (6.137) and Proposition 6.3.12 we obtain[4]

$$\begin{aligned} \tilde{\mathbb{E}}\, h(\mathbf{r}_t\hat{\mathbf{U}}, \mathbf{r}_t\tilde{W}) \left[M(1, \hat{\mathbf{U}}, \hat{\mathbf{U}})_{s,t}\right] &= 0, \\ \tilde{\mathbb{E}}\, h(\mathbf{r}_t\hat{\mathbf{U}}, \mathbf{r}_t\tilde{W}) \left[\left[M(1, \hat{\mathbf{U}}, \hat{\mathbf{U}})^2\right]_{s,t} - N(1, \hat{\mathbf{U}})_{s,t}\right] &= 0, \\ \tilde{\mathbb{E}}\, h(\mathbf{r}_t\hat{\mathbf{U}}, \mathbf{r}_t\tilde{W}) \left[\left[M(1, \hat{\mathbf{U}}, \hat{\mathbf{U}})\tilde{\beta}_k\right]_{s,t} - N(1, \hat{\mathbf{U}})_{s,t}\right] &= 0. \end{aligned} \tag{6.144}$$

where the $\tilde{\mathbb{P}}$-a.s. limit process satisfies

$$\begin{aligned} M(1, \hat{\mathbf{U}}, \hat{\mathbf{U}})_t &= \langle \hat{\mathbf{U}}(t), \boldsymbol{\phi}\rangle - \langle \hat{\mathbf{U}}(0), \boldsymbol{\phi}\rangle - \int_0^t \langle \hat{\mathbf{U}} \otimes \hat{\mathbf{U}} - \nu\nabla\hat{\mathbf{U}}, \nabla\boldsymbol{\phi}\rangle \mathrm{d}r \\ &= \left\langle \hat{\mathbf{U}}_h(t), \boldsymbol{\phi}_h\right\rangle_h - \left\langle \hat{\mathbf{U}}_h(0), \boldsymbol{\phi}_h\right\rangle_h - \int_0^t \left\langle \hat{\mathbf{U}}_h \otimes \hat{\mathbf{U}}_h - \nu\nabla_h\hat{\mathbf{U}}_h\,, \nabla_h\boldsymbol{\phi}_h\right\rangle_h \mathrm{d}r. \end{aligned}$$

Eq. (6.144) implies that $M(1, \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h)$ is an $(\tilde{\mathscr{F}}_t)$-martingale with quadratic and cross variations given by

$$\langle\langle M(1, \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h)\rangle\rangle = N(1, \hat{\mathbf{U}}_h), \quad \langle\langle M(1, \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h), \tilde{\beta}_k\rangle\rangle = N_k(1, \hat{\mathbf{U}}_h)$$

respectively and thus we obtain

$$\left\langle\left\langle M(1, \hat{\mathbf{U}}_h, \hat{\mathbf{U}}_h) - \int_0^{\cdot} \langle \Phi(1, \hat{\mathbf{U}}_h)\mathrm{d}\tilde{W}, \phi\rangle \right\rangle\right\rangle = 0.$$

It therefore follows that $(6.4)_1$ is satisfied in the distributional sense.

The same can be done for $((\tilde{\Omega}, \tilde{\mathscr{F}}_t, (\tilde{\mathscr{F}}_t), \tilde{\mathbb{P}}), \check{\mathbf{U}}, \tilde{W})$. □

6.5.3 Pathwise solvability of the limit problem

Proof of Theorem 6.2.14. With Theorem 6.3.14 in hand, we can now use the assumption on the initial law and Theorem 6.2.12 to get that $\tilde{\mathbb{P}}$-a.s., $\hat{\mathbf{U}}_h(0) = \check{\mathbf{U}}_h(0) =$

[4] We denote by $M(\cdot)_{s,t}$, the difference $M(\cdot)_t - M(\cdot)_s$ and similarly for the other functionals. Also $\mathbf{r}_t$ is a continuous map that restrict functions to time t whereas h is a continuous function. See (4.109) in Chapter 4.

$\mathbf{U}_{h,0}$ and as such, the pair of solutions $\hat{\mathbf{U}}_h$ and $\check{\mathbf{U}}_h$ coincide $\tilde{\mathbb{P}}$-a.s. with law

$$\mu\left(\left(\hat{\varrho}, \hat{\mathbf{U}}, \hat{\mathbf{m}}, \check{\varrho}, \check{\mathbf{U}}, \check{\mathbf{m}}\right) \in \chi \,:\, (\hat{\varrho}, \hat{\mathbf{U}}, \hat{\mathbf{m}}) = (\check{\varrho}, \check{\mathbf{U}}, \check{\mathbf{m}})\right)$$
$$= \tilde{\mathbb{P}}\left((\hat{\varrho}, \hat{\mathbf{U}}, \hat{\mathbf{m}}) = (\check{\varrho}, \check{\mathbf{U}}, \check{\mathbf{m}})\right) = \tilde{\mathbb{P}}\left(\hat{\mathbf{U}}_h = \check{\mathbf{U}}_h\right) = 1.$$

We can now use the generalization to quasi-Polish spaces of the Gyöngy–Krylov characterization of convergence given in [15, Theorem 2.10.3] to show that the original sequence $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \ulcorner \mathcal{P}\,(\varrho_\varepsilon \mathbf{u}_\varepsilon)\urcorner)$ defined on the original probability space $(\Omega, \mathscr{F}, \mathbb{P})$ converges in probability to some random variables $(\varrho, \mathbf{U}_h, \mathbf{m}_h)$ in the topology of $\chi_\varrho \times \chi_{\mathbf{u}} \times \chi_{\ulcorner \varrho \mathbf{u} \urcorner}$.

We can now repeat Section 6.3.13 for $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \ulcorner \mathcal{P}\,(\varrho_\varepsilon \mathbf{u}_\varepsilon)\urcorner)$ and finally get that $\mathbf{U} = \mathbf{U}_h$ is a pathwise solution of (6.4) according to Definition 6.2.11. This repetition is comparatively simpler since we are dealing with the original sequence. As a consequence, it suffices to take only one Wiener process in Lemma 6.3.15 for example.

$\square$

Chapter 7

Published Papers

D. Breit and P. R. Mensah : Stochastic compressible Euler equations and inviscid limits. *arXiv preprint arXiv:1802.07186*, (2018).

P. R. Mensah : A multi-scale limit of a randomly forced rotating 3-D compressible fluid. *arXiv preprint arXiv:1801.09649*, (2018).

P. R. Mensah : Existence of martingale solutions and the incompressible limit for stochastic compressible flows on the whole space. *Ann. Mat. Pura Appl. (4)*, 196(6):2105–2133, (2017).

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